CRITICAL PHENOMENA IN MAGNETS IN A RANDOM MAGNETIC FIELD

by

YUDHVIR SINGH PARMAR

DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

September, 1993

CRITICAL PHENOMENA IN MAGNETS IN A RANDOM MAGNETIC FIELD

A Thesis Submitted
in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

by
Yudhvir Singh Parmar

to the
DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
SEPTEMBER 1993

CERTIFICATE

It is certified that the work contained in this thesis entitled *"Critical Phenomena in Magnets in a Random Magnetic Field"*, by *"Yudhvir Singh Parmar"*, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

J. K. Bhattacharjee
Professor
Department of Physics
Indian Institute of Technology
Kanpur

September 1993

Synopsis

The magnetic systems in the presence of a random magnetic field are good representations of many impure systems. The applicability of the model goes well beyond ferromagnetic systems. The model well describes any solid state system with quenched impurities or lattice defects which locally break the symmetry between the ordered states. This is a general rule and applies to many systems in magnetism, ferroelectricity and surface physics etc. [2].

The random magnetic fields have a marked effect on the second order phase transitions. It is now well established that the lower critical dimension of the random field Ising model is two and that of systems with continuous symmetry is four. It is also clear that as far as critical behaviour is concerned, the random field fluctuations completely dominate the thermal fluctuations. This has led to the general consensus that the critical behaviour is controlled by the zero temperature fixed point. The purpose of this thesis is to show the relevence and utility of zero temperature fixed point for both statics and dynamics of the random field systems.

The thesis is divided into six chapters. The first chapter is an introduction where we review the existing literature. In the second chapter we introduce the phenomenology [16] of the zero temperature fixed point. It is shown that there are three independent exponents in contrast to two for the conventional fixed point. The additional exponent is related to the renormalisation of the random field or equivalently the renormalisation of the temperature.

In the third chapter the replicated approach is combined with decimation technique to set up renormalisation group flows for the random field Ising model (RFIM), which are Migdal–Kadanoff like for dimension $d \geq 2$. Considering the RFIM in $d = 2 + \epsilon$, we find the critical exponents to first order in ϵ. Our results are in agreement with $d \to d - 2 + \eta$ rule , according to which the critical behaviour of a d-dimensional RFIM is identical to the behaviour of pure system in $d - 2 + \eta$ dimension. In $d = 2$, the correlation length $\xi \sim e^{a\frac{J}{H_R}}$ where J and H_R are the spin coupling and random field strength respectively, and a is an arbitrary constant.

The system with continuous symmetry are considered in chapter four. The standard decimation is combined with with Migdal– Kadanoff bond moving approximation to set renormalisation flows. Considering the system in $d = 4 + \epsilon$, the critical exponents are evaluated to first order in ϵ. The exact inequality $\bar{\eta} \leq 2\eta$ due to Schwartz and Soffer [15] is satisfied as equality. This would also imply that the d-dependent hyperscaling law $d\nu = 2 - \alpha$ is modified to $(d - 2 + \eta)\nu = 2 - \alpha$. However, there is no exact dimensional reduction. In $d = 4$ the correlation length $\xi \sim e^{a\frac{J^2}{H_R^2}}$.

In chapter five we study the critical dynamics of the RFIM. There have been two approaches to critical dynamics namely (i) conventional dynamics where the relaxation time $\tau \sim \xi^z$ and (ii) the activated dynamics where the relaxation time $\tau \sim e^{\xi^\theta}$. Recent experiments over a long range of frequencies have revealed that that the critical dynamics is of the activated type but the measured value of the dynamical exponent θ is much smaller than the theoretical expectations. We show that the careful application of dynamical scaling and the zero temperature fixed point leads to an ever present 'correction to scaling' which causes the relevent exponent to differ so strongly from the theoretical expectations.

The last chapter is devoted to one dimensional models in a random field. In section A, we consider the Ginzburg–Landau model in a random magnetic field. Using the replicated hamiltonian it is possible to reduce the problem of calculating the free

energy to that of calculating the the ground state energy of a quantum mechanical problem. The free energy is lowered and specific heat is raised beyond the respective zero field value. In section B, we consider the RFIM in one dimension. Considering the n-times replicated hamiltonian we concentrate on estimating the the largest eigenvalue of the corresponding $2^n \times 2^n$ transfer matrix by using the variational principle. From the estimated largest eigenvalue which permits analytical continuation, we find the free energy for all temperatures. In section C, we take the limit $n \to \infty$ on the replicated hamiltonian of RFIM. The ensuing model is an anisotropic Ising model in two dimensions which has short range interactions in one dimension and weak long range interactions in the other. The model has been exactly solved.

Acknowledgements

Firstly I would like to thank my thesis supervisor Dr. J. K. Bhattacharjee. Without his able guidance and persistent encouragement in every phase of this research work, this thesis would not have taken its present form in shape and content. His valuable contribution is gratefully acknowledged. I would also like to thank Dr. K. Banerjee and Dr. D. Chowdhury for taking a keen interest in my work.

To acknowledge the help and affection of all the people known to me – friends, colleagues and departmental staff – would perhaps be impossible, nor would it be in any way a measure of my gratitude for them. However, I would like thank Sujay for helping me with my thesis typing.

Y. S. Parmar

List Of Publications

1. Y. S. Parmar and J. K. Bhattacharjee **Renormalisation group for the random field Ising Model** *Phy. Rev. B - 46*, 1216, 1992

2. Y. S. Parmar and J. K. Bhattacharjee **One dimensional Ginzburg Landau Model in a Random Magnetic Field** *Phy. Rev. B - 45*, 814, 1992

3. Y. S. Parmar **Renormalisation group for the Heisenberg Magnet in a random field – the Zero temperature Fixed point** submitted for publication to *Europhysics Letters*

4. Y. S. Parmar and J. K. Bhattacharjee **Dynamic Scaling, Zero temperature fixed point and the random field Ising Model** submitted for publication to *Phy. Rev. B*

Contents

Chapter 1

Introduction

The presence of impurities and lattice defects in real materials is an everyday fact of life. It is therefore important to assess their relevence and to study the effects of the same on the results established in pure cases. The theoretical attempts to study the effects of impurities on various phenomena began many years ago but only recently has it become possible to develop the necessary theoretical tools to systematically analyse the important features. The typical problems include the motion of electron in disordered solids, the problem of percolation, spin glasses, etc. [42]

The presence of impurities have marked effects on the critical phenomena for the following reasons. The presence of impurities in materials is equivalent to switching on a small perturbation. The response of the system to small perturbations is characterized by various susceptibilities and correlation functions which become very large near the critical point. Indeed, they are singular functions of temperatures. Therefore even a small amount of impurities can have significant effects. The critical exponents which describe the critical behaviour of various thermodynamic quantities may get modified. Even the critical point may disappear, leading to destruction of long range order. These effects are nontrivial and are still imperfectly understood.

We shall mainly be interested in random field phase transitions in which the field

conjugate to order parameter is random (See Ref. ([1]-[3]) for recent review). The random field problem is described by the hamiltonian

$$H = -J\sum_{ij}\vec{S}_i.\vec{S}_j - \sum_i \vec{h}_i.\vec{S}_i \tag{1.1}$$

where $\vec{S}_i$ and $\vec{h}_i$ are n-component vectors with $|\vec{S}_i| = 1$. The random field $\vec{h}_i$ has the Gaussian distribution.

$$\langle S_i^\alpha S_j^\beta\rangle = \delta^{\alpha\beta}\delta_{ij}H_R^2 \tag{1.2}$$

The case $n = 1$ corresponds to the random field Ising model (RFIM) while $n \geq 2$ describes the systems with continuous symmetry.

Ginzburg - Landau version of the model is also very useful in analysis of the critical behaviour. In place of discrete spins and discrete lattice sites one has a continuous, coarse grained spin variable $\vec{\phi}(r)$ and the hamiltonian is now

$$H = \int d^dx \left[\frac{1}{2}r\vec{\phi}.\vec{\phi} + \frac{u}{4}(\vec{\phi}.\vec{\phi})^2 + (\nabla\vec{\phi})^2 - \vec{h}(r).\vec{\phi}(r)\right] \tag{1.3}$$

where

$$\vec{\phi}.\vec{\phi} = \sum_{i=1}^{n}\phi^2(x)$$

$$\nabla\vec{\phi}.\nabla\vec{\phi} = \sum_{\alpha=1}^{d}\sum_{i=1}^{n}(\frac{\partial\phi_i}{\partial x_\alpha})^2$$

and the random field distribution is

$$\langle h_i(r)h_j(r'\rangle = H_R^2\delta(r-r')\delta_{ij} \; .$$

The random field model is a good representation of many impure materials. The applicability of the model goes well beyond mere ferromagnetic systems. The model well describes any solid state system with frozen impurities and point defects which locally break the symmetry between ordered states. There are many systems in magnetism, ferroelectricity and surface physics where this situation arises [2]. From

the experimental point of view, Ising antiferromagnet (AF) in an external field H_e is of great significance. Fishman etc. [45] showed that the AF in the presence of external field can be mapped on to the RFIM. The AF allows a systematic experimental study. The effective random field H_R vanishes with H_e and can be varied around zero just by changing H_e. This allows one to see how the relevent quantities change with H_e.

The field has seen enormous activity in last one and a half decade since the pioneering work of Imry and Ma (IM) [4] who pointed out that even small random field can have drastic effects on the large scale behavior of the system. Introducing basic theoretical concepts, they showed that long range order established in pure cases is destroyed by arbitrary small random fields below the lower critical dimension which is two for the Ising like systems and four for the systems with continuous symmetry.

The salient feature of the random field problem is the conflict between the two terms in the hamiltonian. The spin coupling J favours allignment of spins while random field strength H_R favours disorder. If $H_R \gg J$, then the system is expected to be disordered at low temperatures. However, interesting possibility arises when random field is small.

The usual procedure to determine the lower critical dimension is to consider the stability of completely ferromagnetic state to the formation of domains of opposite spins. IM proof consisted in showing that the formation of ill-oriented domains is always energetically favourable for large enough domain size below the lower critical at $T = 0$. The argument automatically holds for finite T because thermal fluctuations favour disorder.]?

IM proof is simple and throws light at the heart of the problem,so we shall go through it in some detail. Consider a bubble (domain) of linear size L in a sea of down spins. Let the spins inside the bubble point in the up direction. The domain wall energy for Ising like systems does like L^{d-1} as the number of spins that are to be flipped is proportional to L^{d-1}. For systems with continuous symmetry one

can decrease domain wall energy cost by avoiding a sharp flip at the boundary by continuously changing the orientation of the spins along the whole domain. Simple arguments show that the energy cost in this case goes like L^{d-2}. The random field energy gain is the sum of L^d random variables. The typical value of the random field energy is therefore $\sim H_R L^{d/2}$ with positive or negative sign but it is always possible to find a region enclosing an arbitrary point such that $E_{RF} \geq 0$. So the total energy cost for a bubble of size L is

$$\begin{aligned} E(L) &= JL^{d-1} - H_R L^{d/2} && \text{for Ising like systems} \\ &= JL^{d-2} - H_R L^{d/2} && \text{for systems with continuous symmetry} \end{aligned}$$

For large enough L energy cost can always be made negative for $d < 2$ and $d < 4$ for Ising like systems and systems with continuous symmetry respectively, no matter how small the random field is. Thus the ferromagnetic state becomes unstable to domain formation for $d < 2$ for Ising systems and for $d <$ 4for continuous systems.

Using renormalisation group arguments, IM also verified that the upper critical dimension, the dimension above which the exponents are those of mean field theory, is six. They suggested ϵ expansion near $d = 6$ for $d = 6-\epsilon$ and carried out the calculation to first order in ϵ. Grinstein has [5] considered the $O(\epsilon^2)$ terms and found that d is replaced by $d-2$ in the exponents and hyperscaling relation of the pure system. This led to the astonishing idea of dimensional reduction whereby critical behaviour of d-dimensional random field system is claimed to be same as that of pure system in $d-2$ dimension. Aharony etc. [6] attempted to prove that dimensional reduction holds to all orders in ϵ. The complete calculation has been first done correctly by Young [7]. Parisi and Sourla [8] avoided the combinatorial difficulty encountered in this approach by introducing supersymmetric formulation. This enabled them to present the proof in a much more elegant way.

The dimensionality reduction would imply that lower critical dimension of RFIM

is three and that of systems with continuous symmetry is four. For systems with continuous symmetry this agrees with IM predictions but is in conflict with their results for RFIM. Since the IM predictions are based on handwaving arguments, a reconciliation was sought by arguing that IM domain wall arguments give a lower bound and there might be a different mechanism which renders ferromagnetic state unstable to formation of ill-oriented domains. Attempts were made, [9] and [11], to obtain $d = 3$ as the lower critical dimension and do the ϵ expansion for $d = 3+\epsilon$ for R-F interface model. It was later pointed out [10] that the qualitative aspect of this work has some difficulties. Later work [20]-[21] involving a very plausible $R-G$ procedure on $R-F$ interface re-established $d_l = 2$. The controversy was finally settled when rigorous analysis [45] proved that the three dimensional RFIM exhibits long range order at zero temperature and small disorder, disproving the dimensionality reduction predictions.

This leads us to question the validity of the analysis which imply dimensionality reduction $d \rightarrow d-2$. The derivation is based on the assumption that only a selected class of diagrams which are most divergent near the assumed transition point are relevant in evaluating the critical exponents [6] . These diagrams are called the tree diagrams and the approximation amounts to neglecting the thermal fluctuations. This is equivalent to solving the local mean-field equations. The difficulty with dimensional reduction arguments becomes transparent in Parisi–Sourlas [8] approach. They assume that the local mean field equations can be solved and order parameter can be expressed as a function of the random field. After this they average over the random field configurations. However, they implicitly assume only one solution to the local mean- field equations. This assumption (as was later pointed by them) is true above the transition temperature of pure mean-field problem but breaks down below this temperature where multiple solutions are possible. The transition temperature in presence of random field lies below the transition temperature of the pure problem.

Hence, the implicit assumption which leads to dimensionality reduction breaks down. The same will hold in the other approach which in essence is equivalent to Parisi–Sourlas approach.

Schwartz [14] pursued a novel and an interesting idea of constructing an equivalent annealed model whose statistical properties are identical to the random field Ising model near the critical point. The ensuing model is complicated and has long range correlations. But Schwartz claimed that considering the leading behaviour of the correlation function, it can be shown that the random field Ising model has the same critical behaviour as the pure Ising model in $d' = d-2+\eta(d)$. Schwartz indicated that this result may be generalised to arbitrary dimensionality of the order parameter. It is interesting to note that Schwartz reduction gives lower critical dimension in agreement with Imry–Ma result.

The critical dynamics of the system is also drastically modified from dynamics of the pure system. Experiments [24]-[28] performed on the diluted Ising antiferromagnets (AF) in uniform field - physical realisation of RFIM - reveal quite unusual properties. Since $d_l = 2$, AF should exhibit long range order for low temperature and small external field. But when cooled in non-zero fields, AF do not exhibit any long range order down to very low temperatures even for modest field strength. On the other hand long range order established by cooling in zero field persist under the application of quite large fields. The history dependent behaviour implies that equilibration time is very large. Specifically critical slowing down is so extreme that it exceeds the largest observation time.

The analysis around the upper critical dimension, [30] and [31], indicate that the critical dynamics is of conventional type with power law divergence. But this is not in accord with experimental findings which show that the relaxation times are extremely large. To explain the data in terms of power law divergence would require unphysically large value of dynamical exponent [27], [29]. To account for

the extremely large relaxation times Villain [19] and Fisher [18] have suggested the activated dynamics where equilibration requires jumps between remote energy wells in phase space. The relaxation time, according to Villian and Fisher diverges like $\tau \sim \exp(\xi^{\theta})$ where θ is the exponent that enters the hyperscaling relations. Recent experiments [28] have confirmed that the critical dynamics is of activated type but the exponent θ is quite different from the theoretical expectations.

It is now clear that as far as critical behaviour is concerned, the random field fluctuations completely dominate the thermal fluctuations. This has led to the general consensus that critical behaviour is controlled by the zero temperature fixed point. The main purpose of this thesis is to show the relevence and utility of the zero temperature fixed point for both statics and dynamics of the random field systems. In the next chapter we introduce the phenomenology of the zero temperature fixed point. The fact that there are three independent exponents in contrast to two for the conventional systems is stressed. The additional exponent is related to the renormalization of the random field. In the third chapter replicated approach is combined with standard decimation technique to set up renormalisation group flows for the random field Ising model (RFIM), which are Migdal–Kadanoff like for dimensions $d \geq 2$. Considering the RFIM in $d = 2 + \epsilon$, we find the critical exponents to first order in ϵ.

The systems with continuous symmetry are considered in chapter four. For these systems ($n \geq 2$), both dimensional reduction and IM domain wall arguments yield four as the lower critical dimension. It is thus possible that Parisi-Sourlas reduction holds all the way from six down to four dimensions. Indeed, it was claimed in literature [6] that for $n \geq 2$ to leading order in $\epsilon = d - 4$, the critical behaviour of the random field model is the same as that of pure model in $d = 2 + \epsilon$. Fisher [23],however showed that these calculations are incorrect.The fixed point found by Young [6] in $d = 4 + \epsilon$ was shown to be unstable to random anisotropies. In chapter four, by combining

decimation and Migdal Kadanoff bond moving approximation we find another fixed point in $d = 4 + \epsilon$. Considering the system in $d = 4 + \epsilon$, we find a new set of critical exponents to first order in ϵ.

In chapter five we study the critical dynamics of the RFIM. We show that careful analysis of dynamics at the zero temperature fixed point combines conventional dynamics with the activated dynamics. It is shown that the application of dynamic scaling and zero temperature fixed point leads to an everpresent correction to scaling which causes the exponent θ to differ so strongly from the theoretical expectations. The last chapter is devoted to one dimensional models in a random magnetic field. In section A, we consider the one dimensional Ginzburg–Landau model in a random magnetic field. Using the replicated hamiltonian it is possible to reduce the problem of calculating the free energy to that of calculating the ground state energy of a quantum mechanical problem. In section B, we consider the random field Ising model in one dimension. Considering the n-times replicated hamiltonian we concentrate on estimating the largest eigenvalue of the corresponding $2^n \times 2^n$ transfer matrix by using the variational principle. From the estimated eigenvalue which permits analytical continuation, we find the free energy for all temperatures.

Chapter 2

Zero Temperature Fixed Point

2.1 Renormalisation group analysis

A completely satisfactory renormalisation group treatment of ferromagnets in a random magnetic field is not present although a good understanding of the structure of flow equation exist due the pioneering work of Bray and Moore [16] and somewhat more general exposition of Berker and McKay [17]. We shall closely follow Bray and Moore and a review article by Nattermann and Rujan [3] in our presentation.

The model for ferromagnets in a random field is defined by the hamiltonian

$$H = -\frac{J}{T}\sum_{\langle i,j\rangle} \vec{S}_i.\vec{S}_j - \frac{1}{T}\sum_i \vec{h}_i.\vec{S}_i \tag{2.1}$$

where the random field has a Gaussian distribution

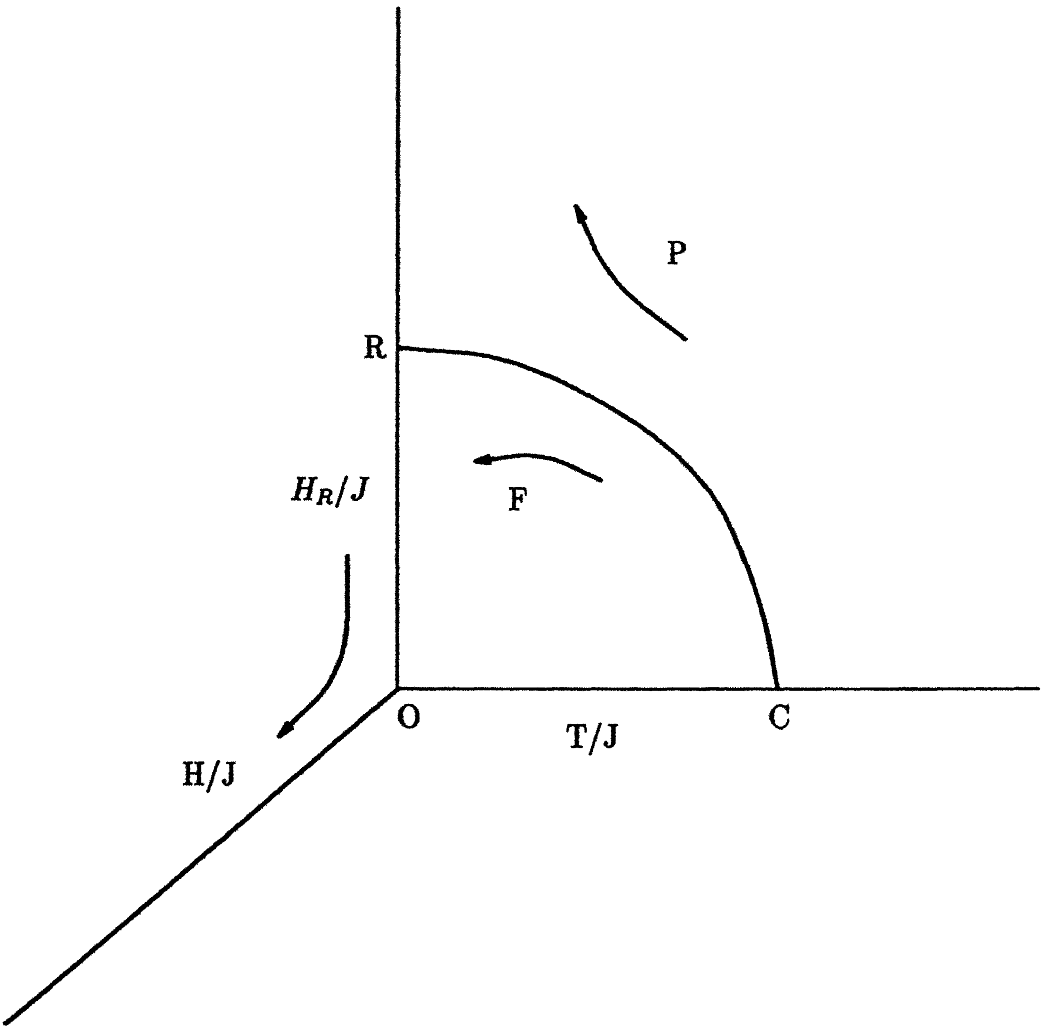
$$\langle h_i^\alpha h_j^\beta \rangle = \delta^{\alpha\beta}\delta_{ij}H_R \tag{2.2}$$

$$\langle h_i^\alpha \rangle = H\delta_{\ \alpha} \tag{2.3}$$

We shall assume an infinitesimal field in direction 1. The indices i, j as usual denote the lattice sites and α, β which refer to the field components run from 1 to n. The

cases $n = 1$, $n = 2$ and $n > 2$ correspond to Ising, X–Y and Heisenberg model respectively.

We assume that equilibrium phase transition remains continuous down to $T = 0$ for Gaussian random fields. The phase diagram expected for this model is shown in figure 1. The existence of three fixed points will be assumed in addition to the trivial, high temperature fixed point.



The three fixed points are

1. A totally unstable fixed point C at $T = T_c$, $\frac{H_R}{J} = H = 0$.

2. A fixed point R at $T = H = 0$ and $\frac{HR}{J} = w^*$ which is unstable in two but stable in in one direction.

3. A totally stable fixed point O at $T = \frac{HR}{J} = H = 0$ which corresponds to low temperature phase. All flows in the ferromagnetic region terminate at this point.

The critical behaviour of the pure system is controlled by the thermal fixed point C. But since random field is a relevent perturbation, flows along the phase boundary approach the zero temperature fixed point R. The fixed point R controls the critical behaviour in presence of the random field.

To calculate the critical exponents we linearize the $R-G$ flow close to the fixed point R. The eigenvalues and eigenvectors of the linearized $R-G$ flow transformation deliver then the critical exponents and the scaling fields. Phenomenological arguments concerning the $R-G$ flow suggest $\frac{T}{J}$, $t = \frac{HR}{J} - w^* + \frac{T}{J}$ (const) and $\frac{H}{J}$ as the scaling fields.

On the basis of this picture one can now derive the scaling laws. From dimensional considerations the singular part of the free has the form

$$F_s = Jf\left(\frac{T}{J}, t, \frac{H}{J}\right) \tag{2.4}$$

Now imagine carrying out an $R-G$ coarse graining transformation, with length scale factor b, corresponding to reduction of degrees of freedom by a scale factor b^d. Introducing exponents y_J, y_h and y_t which describe the scaling of J, H and t respectively, the coarse-grained J', H' and t' are given by

$$\begin{aligned} J' &= Jb^{y_J} \\ t' &= tb^{y_t} \\ H' &= Hb^{y_h} \end{aligned} \tag{2.5}$$

$\beta = \nu(d - y_h)$	$\nu = 1/y_t$
$\gamma = \nu(2y_h - y_J - d)$	$\eta = 2 + d + y_J - 2y_h$
$\delta = \frac{y_h - y_J}{d - y_h}$	$\bar{\eta} = 4 + d - 2y_h$

Table 2.1: Scaling Relations

From equations (2.4) and (2.5) and the fact that partition function is invariant under $R-G$ coarse-graining we obtain the following scaling relations for the free energy density and the correlation length

$$f\left(\frac{T}{J}, t, \frac{H}{J}\right) = b^{y_J - d} f(\frac{T}{J} b^{-y_J}, t b^{y_t}, \frac{H}{J} b^{y_h - y_J}) \tag{2.6}$$

$$\xi\left(\frac{T}{J}, t, \frac{H}{J}\right) = b\xi\left(\frac{T}{J} b^{-y_J}, t b^{y_t}, \frac{H}{J} b^{y_h - y_J}\right) \tag{2.7}$$

The scaling relations and critical exponents follow in the usual way (Stanley 1971,[51]). For detailed analysis, see Ref.[16]. The relations between y_J, y_t and y_h and critical exponents are summarized in the table below

It is to be noted that there are three independent exponents in contrast to two independent exponents in the pure case. The additional exponent appearing in the present case is intimately connected to the fact that we are considering the zero temperature fixed point 'R'. In absence of random field the thermal fixed point 'C' controls the critical behaviour. The analogue of eq.(2.4)is

$$F_s = Jf(t, \frac{h}{J}) \tag{2.8}$$

where $t = \frac{T}{J} - (\frac{T}{J})^*$ and F_s is the singular part of the free energy. At the thermal fixed point 'C', the exponent y_J appearing in eq.(2.6) is zero because J is invariant under the $R-G$ coarse graining. But at the fixed 'R', it is the ratio $\frac{H_R}{J}$ which is

while the coupling J scales up at the fixed point.From eqs.(2.4) and (2.5) it is clear that free energy scales up at the fixed point 'R' if $y_J \geq 0$. This leads to the modified hyperscaling law.

$$(d - y_J)\nu = 2 - \alpha \tag{2.9}$$

The additional exponent $\bar{\eta}$ appearing in Table (2.1) is associated with the scaling of the disconnected part of the correlation function

$$G_{dis} = \overline{\langle S_0 \rangle \langle S_r \rangle} = \frac{1}{r^{d-4+\bar{\eta}}} g_{dis}(r/\xi) \tag{2.10}$$

2.2 Explicit evaluation of critical exponent

Since the critical behaviour is controlled by the zero temperature fixed point 'R' we can evaluate exponents simply by working at $T = 0$. The exponents can be obtained from $R - G$ differential equations of the form

$$\frac{dH_R}{dl} = H_R f_1(w) \tag{2.11}$$

$$\frac{dJ}{dl} = J f_2(w) \tag{2.12}$$

$$\frac{dH}{dl} = H f_3(w) \tag{2.13}$$

where $w = \frac{H_R}{J}$, coresponding to the length scale factor $b = e^l$. In general renormalization group transformation generates new terms in the hamiltonian. We shall assume that these terms are irrelevent in the $R - G$ sense at least near the lower critical dimension.

The fixed point 'R' is obtained from the equation for w. Using eqs. (2.11) and (2.12) we get

$$\frac{dw}{dl} = \frac{d}{dl}\left(\frac{H_R}{J}\right) = w(f_1(w) - f_2(w)) \tag{2.14}$$

At the fixed point

$$\frac{dw}{dl}\Big|_{w=w^*} = o$$

giving

$$f_1(w^*) = f_2(w^*)$$

at the fixed point. Linearizing around the fixed point gives the exponent ν,

$$y_t = \frac{1}{\nu} = w^*(f_1'(w) - f_2'(w)) \tag{2.15}$$

where the primes denote the derivative of $f_i(w)$ with respect to w.

The other exponents y_J and y_h are given by

$$y_h = f_1(w^*) = f_2(w^*) \tag{2.16}$$

$$y_h = f_3(w^*) \tag{2.17}$$

Simple domain-wall arguments due to Imry and Ma suggest that the lower critical dimension of random field Ising model is two and that of magnets with continuous symmetry is four. Near the lower critical dimension the fixed point w^* is going to be small. This allows a perturbative treatment of the functions $f_i(w)$.

In what follows we shall use Migdal–Kadanoff bond moving approximation to valuate critical exponents in $d = d_l + \epsilon$ for the Ising and the Heisenberg magnets in random magnetic field. The Migdal–Kadanoff bond moving renormalization group ransformation was invented by Migdal [39] and reinterpreted using bond moving eas by Kadanoff [41]. In Kadanoff interpretation, bonds are moved so that spins y be eliminated by one dimensional decimation. The bond moving approximation xact for $T \to 0$ and $\frac{H_R}{J} \to 0$. So we expect that in $d = d_l + \epsilon$ when $T = 0$ and $\sim O(\epsilon^\alpha)$, where $\alpha > 0$; our results are exact. The bond moving predictions have y been found to be exact [39] for pure Ising and Heisenberg models near the itical dimension.

Chapter 3

Random–field Ising Model

In this chapter we will consider a Random–field Ising Model. The replicated approach is combined with the standard spin–decimation technique to set up renormalisation group flows which are Migdal–Kadanoff–like for dimensions D greater than or equal to 2.

The partition function of the random field Ising model (RFIM) is

$$Z = \sum_{\{S_i\}} \exp[\frac{J}{T} \sum_{\langle i,j \rangle} S_i S_j + \frac{1}{T} \sum_i h_i S_i] \tag{3.1}$$

where the random field h_i at the site i has a Gaussian distribution

$$\langle h_i h_j \rangle = \delta_{ij} H_R^2 \tag{3.2}$$

We assume zero uniform external field. We shall use n-times replicated system in one dimension which has the partition function (after averaging over the field h_i) (See Appendix 1)

$$\begin{aligned} Z^n &= \sum_{S_i^{(\alpha)}} \exp\left[\frac{J}{T}\sum_{i,\alpha} S_i^\alpha S_{i+1}^\alpha + \frac{H_R^2}{T^2}(\sum S_i^\alpha)^2\right] \\ &= \sum_{S_i^{(\alpha)}} \exp\left[K\sum_{i,\alpha} S_i^\alpha S_{i+1}^\alpha + \Delta \sum_{\alpha\neq\beta} S_i^\alpha S_i^\beta + n\Delta\right] \end{aligned} \tag{3.3}$$

where $K = \frac{J}{T}$ and $\Delta = \frac{H_R^2}{T^2}$. α and β are the replica indices and they run from 1 to n. The RFIM corresponds to $n = 0$ in the sense that free energy which is proportional to $\langle \ln Z(h) \rangle$, can be obtained from $lim_{n \to 0} \left[\frac{\langle Z^n \rangle - 1}{n}\right]$. The idea at this point is to carry out the usual renormalization group elimination of degrees of freedom by doing the trace over every alternate spin - in this case over the set S^α - and thus obtain a flow $K' = f(K, \Delta, n)$ and $\Delta' = g(K, \Delta, n)$ and take the limit $n \to 0$ which gives the flow for the RFIM. Indeed, the Taylor expansion about $n = 0$ is

$$K' = f(K, \Delta, n) = K_0' + nK_1' + o(n^2) \tag{3.4}$$

$$\Delta' = \Delta_0' + n\Delta_n' + o(n^2) \tag{3.5}$$

Terms proportional to n and higher order in n on the right hand side of eqs. (3.4) and (3.5) contribute $o(n^2)$ corrections to the partition function. The free energy is given by

$$lim_{n \to 0} \langle \frac{Z^n - 1}{n} \rangle$$

and hence is unchanged by these corrections. Therefore the renormalized K' and Δ' are just K_0' and Δ_0' respectively.

We begin by decimating the Ising chain in a random field. The site index i in eq.(3.3) is a single integer, and we have spins $S_1^\alpha, S_2^\alpha, \ldots$ etc. on various sites. The renormalization group procedure would be to do the trace over alternate spins $S_2^\alpha, S_4^\alpha, S_6^\alpha, \ldots$ etc.. If we consider sites 1, 2, and 3, then the spins at site 2 will be removed and the partition expressible in terms of S_1^α and S_3^α. We thus require the relevant bit of Z^n, which we write as

$$Z_n' = \sum_{\{S_2^\alpha\}} \exp\left[K \sum_\alpha S_1^\alpha S_2^\alpha + S_2^\alpha S_3^\alpha\right] \exp\left[\sum_{\alpha \neq \beta} \frac{\Delta}{2}(S_1^\alpha S_1^\beta + 2S_2^\alpha S_2^\beta + S_3^\alpha S_3^\beta)\right] \tag{3.6}$$

There are n spins at site 2 which means that we have to sum over 2^n configurations.

Summing over all these configurations leads to

$$\begin{aligned}
Z'_n &= 2\exp\left[\frac{\Delta}{2}\sum_{\alpha\neq\beta}(S_1^\alpha S_1^\beta + S_3^\alpha S_3^\beta)\right] \\
&\left[\cosh K\sum_{\alpha=1}^{n}\phi^\alpha + e^{-4(n-1)\Delta}\sum_{m_1}^{n}\cosh K\{\sum_\alpha \phi^\alpha - 2\phi^{m_1}\} + \ldots\right. \\
&\ldots + \frac{e^{-4(N-1)(n+1-N)\Delta}}{(N-1)!}\sum_{m_1\neq m_2\neq\ldots\neq m_N}\cosh K\{\sum_\alpha \phi^\alpha - 2\phi^{m_1} - 2\phi^{m_2} - \cdots - 2\phi^{m_N}\} \\
&\left. + \cdots\right] e^{n(n-1)\Delta} \qquad (3.7)
\end{aligned}$$

where

$$\phi^\alpha = S_1^\alpha + S_3^\alpha.$$

One can obtain eq.(3.7) by explicitly working out the cases $n = 2, 3\ldots$ etc. and then generalizing to arbitrary n after observing the pattern that emerges.

To implement the renormalization- group program, we want to write eq.(3.7) as

$$Z'_n = \exp\left[\sum_\alpha K' S_1^\alpha S_3^\alpha + \sum_{\alpha\neq\beta}\frac{\Delta'}{2}(S_1^\alpha S_1^\beta + S_3^\alpha S_3^\beta) + a\right] \qquad (3.8)$$

To find K', Δ', and a as functions of K, Δ, and n, we evaluate eq.(3.7) and eq.(3.8) for three specific situations :

1. each of $\{S_1^\alpha\}$ and each of $\{S_3^\alpha\}$up,

2. each of $\{S_1^\alpha\}$ up and each of $\{S_3^\alpha\}$ down,

3. $S_1^{(1)}, S_1^{(2)}, \ldots\ldots S_1^{(n-1)}$ up and $S_1^{(n)}$ down, while $S_3^{(1)}, S_3^{(2)}, \ldots\ldots S_3^{(n-1)}$ up and $S_3^{(n)}$,

The three cases lead to

$$\begin{aligned}
e^{[nk\ + n(n-1)\Delta\ + a]} &= 2e^{n(n-1)\Delta}[\cosh(2nk) +^n C_1 e^{-4(n-1)\Delta}\cosh\{2(n-2)K\}\ldots \\
&\quad +^n C_{N-1} e^{-4(N-1)\{n-(n-1)\}\Delta}\cosh[2\{n-2(N-1)\}K] + \cdots] \\
&= 2e^{n(n-1)\Delta} A(n) \qquad (3.9)
\end{aligned}$$

$$\begin{aligned} e^{[-nk'+n(n-1)\Delta'+a]} &= 2e^{n(n-1)\Delta}[1+^{n}C_1 e^{-4(n-1)\Delta}\dots \\ &\quad +^{n}C_{N-1}e^{-4(N-1)\{n-(n-1)\}\Delta}+\cdots] \\ &= 2e^{n(n-1)\Delta}\,B(n) \end{aligned} \tag{3.10}$$

$$\begin{aligned} &e^{[nK'+(n-1)(n-4)\Delta'+a]} = 2e^{(n-1)(n-4)\Delta}\,[\cosh\{2(n-2)K\}+ \\ &\{\cosh 2nK+(n-1)\cosh[2(n-4)K]\}e^{-4(N-1)\{n-(N-1)\}\Delta}+\dots+ \\ &\{^{n-1}C_{N-1}\cosh[2(n-2)K]+^{n-1}C_{N-2}\cosh[2(n-2(N-2))K]\}e^{-4(N-1)[n-(N-1)]\Delta}+\dots \\ &= 2e^{(n-1)(n-2)\Delta}\,C(n) \end{aligned} \tag{3.11}$$

Considering eqs.(3.9) and (3.10) and dividing one by the other and taking the logarithm of both the sides we end up with

$$\begin{aligned} 2nK' &= \ln[\cosh 2nK+^{n}C_1 e^{-4(n-1)\Delta}\cosh 2(n-2)K+\cdots+ \\ &\quad ^{n}C_{N-1}e^{-4(N-1)\{n-(N-1)\}\Delta}\cosh 2\{n-2(N-1)\}K+\cdots] \\ &\quad -\ln[1+^{n}C_1 e^{-4(n-1)\Delta}+\cdots+ \\ &\quad ^{n}C_{N-1}e^{-4(N-1)\{n-(N-1)\}\Delta}+\cdots] \end{aligned} \tag{3.12}$$

Eq.(3.12) admits analytical continuation. Expanding R. H. S. of eq. (3.12) in power of n we get

$$\begin{aligned} 2nK' &= \ln[1+n(e^{4\Delta}\cosh 4K-\frac{1}{2}e^{16\Delta}\cosh 8K+ \\ &\quad \cdots+\frac{(-1)^N}{N-1}e^{4(N-1)^2\Delta}\cosh 4(n-1)K+\cdots)+o(n^2)] \\ &\quad -\ln[1+n(e^{4\Delta}-\frac{1}{2}e^{16\Delta}+\cdots \\ &\quad +\frac{(-1)^N}{N-1}e^{4(N-1)^2\Delta}+\cdots)+o(n^2)] \\ &= n(e^{4\Delta}\cosh 4K-\frac{1}{2}e^{16\Delta}\cosh 8K+\cdots+\frac{(-1)^N}{N-1}e^{4(N-1)^2\Delta}\cosh 4(n-1)K+\cdots) \end{aligned}$$

$$-n(e^{4\Delta} - \frac{1}{2}e^{16\Delta} + \cdots + \frac{(-1)^N}{N-1}e^{4(N-1)^2\Delta} + \cdots) + o(n^2) \tag{3.13}$$

Taking the limit $n \to 0$ in eq.(3.13) leads to

$$2K' = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} e^{4m^2\Delta} \left[\cosh(4mK) - 1\right] \tag{3.14}$$

It is straightforward to check that for $\Delta = 0$ eq.(3.14) reduces to the well known recursion relation for one dimensional Ising model.

To find the recursion relation for Δ, we divide eq.(3) by eq.(3.9) to get

$$e^{-4(n-1)\Delta'} = e^{-4(n-1)\Delta} \frac{C(n)}{A(n)} \tag{3.15}$$

At this stage we set $n = 0$ in eq.(3.15) and obtain

$$e^{4\Delta'} = e^{4\Delta} \frac{C(0)}{A(0)} \tag{3.16}$$

From the expressions for $A(n)$ and $C(n)$ we find

$$A(0) = 1$$

and

$$C(0) = \sum_{m=1} (-1)^{m+1} \{e^{4(m-1)^2\Delta} \cosh 4mK + e^{4m^2\Delta} \cosh[4(m-1)K]\}$$

We may now write eq.(3.16) as

$$e^{4(\Delta'-\Delta)} = \sum_{m=1} (-1)^{m+1} \{e^{4(m-1)^2\Delta} \cosh 4mK + e^{4m^2\Delta} \cosh[4(m-1)K]\} \tag{3.17}$$

Eqs.(3.14) and (3.17) are the renormalization group flows for the one dimensional RFIM.

The series appearing in eqs.(3.14) and (3.17) are divergent and as such are not manageable. Using the identity

$$e^{4m^2\Delta} = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-p^2 + 4pm\sqrt{\Delta}} dp \tag{3.18}$$

we can recast eq.(3.14) in the form

$$
\begin{aligned}
K' &= \frac{1}{4\pi^{1/2}}\int_{-\infty}^{\infty} dpe^{-p^2}\left[\sum_{m=1}\frac{(-1)^{m+1}}{m}\{e^{4mK+4mp\sqrt{\Delta}}+e^{-4mK+4mp\sqrt{\Delta}}-2e^{4pm\sqrt{\Delta}}\}\right] \\
&= \frac{1}{4\pi^{1/2}}\int_{-\infty}^{\infty} dpe^{-p^2}\left[\ln(1+e^{4p\sqrt{\Delta}+4K})+\ln(1+e^{4p\sqrt{\Delta}-4K})-2\ln(1+e^{4p\sqrt{\Delta}})\right] \\
&= \frac{1}{4\pi^{1/2}}\int_{-\infty}^{\infty} dpe^{-p^2}\ln\left[\frac{(1+e^{4p\sqrt{\Delta}+4K})(1+e^{4p\sqrt{\Delta}-4K})}{(1+e^{4p\sqrt{\Delta}})^2}\right] = f(K,\Delta) \qquad (3.19)
\end{aligned}
$$

In arriving at the final expression we have used the identity

$$\sum_{n=1}^{\infty}\frac{x^n}{n}(-1)^{n+1} = \ln(1+x)$$

In an identical fashion, by substituting eq.(3.18) in eq.(3.17) we find

$$
\begin{aligned}
e^{4(\Delta'-\Delta)} &= \frac{1}{2\pi^{1/2}}\int_{-\infty}^{\infty}\sum_{m=1}(-1)^{m+1}e^{-p^2}\left[e^{4(m-1)\sqrt{\Delta}p+4mK}+e^{4(m-1)\sqrt{\Delta}p-4mK}+\right. \\
&\qquad \left. e^{4m\sqrt{\Delta}p+4(m-1)K}+e^{4m\sqrt{\Delta}p-4(m-1)K}\right] \qquad (3.20)
\end{aligned}
$$

Now, using the identity

$$\sum_{n=1} x^n(-1)^{n+1} = \frac{x}{1+x}$$

we get

$$e^{4(\Delta'-\Delta)} = \int_{-\infty}^{\infty} f_1(K,\Delta)e^{-p^2}dp + \int_{-\infty}^{\infty} f_2(K,\Delta)e^{-p^2}dp \qquad (3.21)$$

where

$f_1(K,\Delta) = \frac{1}{2\sqrt{\pi}}\left[\frac{e^{4K}}{1+e^{4K+4\sqrt{\Delta}p}}+\frac{e^{-4K}}{1+e^{-4K+4\sqrt{\Delta}p}}\right]$

and

$f_2(K,\Delta) = \frac{1}{2\sqrt{\pi}}e^{4\sqrt{\Delta}p}\left[\frac{1}{1+e^{4K+4\sqrt{\Delta}p}}+\frac{1}{1+e^{-4K+4\sqrt{\Delta}p}}\right]$.

We shall work in the regime where $T \to 0$ and $w = \frac{H_R}{J}$ is small. In terms of K and Δ this implies $K \to 0, \Delta \to 0$ such that $\frac{\sqrt{\Delta}}{K} = w$ is small.

In the limit of large K and Δ eq.(3.19) reduces to

$$K' = \frac{1}{\sqrt{\pi}}\int_{1/w}^{\infty} K e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{0}^{1/w}(K - wpK)e^{-p^2}\,dp + \frac{1}{\sqrt{\pi}}\int_{-1/w}^{0}(K + wpK)e^{-p^2}\,dp$$

which for small w reduces to

$$K' = K(1 - \frac{1}{\sqrt{\pi}}w) + (\text{exp. small corrections of the type } e^{-1/w^2}) \tag{3.22}$$

The random field has the recursion formula given by eq.(3.21). In the limit $K \to \infty$, the right hand side of eq.(3.21) is dominated by the contribution from first term of $f_1(K, \Delta)$ and the second term of $f_2(K, \Delta)$. The integration now leads to the recursion

$$\Delta' = 2\Delta \tag{3.23}$$

which shows the relevance of Δ and thus at $T = 0$, the pure fixed point in one dimension is unstable. It should be to be noted that, for small w, the recursion to the above recursion involve terms such as e^{-1/w^2}, which we can safely ignore.

We now use the Migdal-Kadanoff bond moving approximation to extend the results of eqs.(3.22) and (3.23) to higher dimensions. The approximation assumes the coupling K appearing in one dimension to be changed to $2^{d-1}K$ in d dimensions, corresponding to accumulated bond strength in a $(d-1)$ hypersurface - the spins on which are removed in a decimation procedure. It is valid at low temperature and small random field where an ordered state is assumed so that a simple addition of individual bonds can be envisaged. The extent to which the single site interaction $\Delta \sum_{\alpha,\beta} S_i^\alpha S_i^\beta$ should be moved in higher dimensions before decimations is ambiguous. We follow Migdal [39] in associating a combined interaction $K \sum_\alpha S_i^\alpha S_j^\alpha + \frac{\Delta}{2d}\sum_{\alpha,\beta}(S_i^\alpha S_j^\beta)$ with each bond and moving both the spin coupling K and the random field strength Δ. Thus the recursion relations eqs.(3.19) and (3.21) become

$$K' = f(2^{d-1}K, 2^{d-1}\Delta) \tag{3.24}$$

$$e^{4\Delta'} = e^{2^{-1}(4\Delta)} \left[\int_{-\infty}^{\infty} f_1(2^{d-1}K, 2^{d-1}\Delta) e^{-p^2} dp + \int_{-\infty}^{\infty} f_2(2^{d-1}K, 2^{d-1}\Delta) e^{-p^2} dp \right] \tag{3.25}$$

where f_1 and f_2 are the same functions as before.

We now demonstrate that, for $w \ll 1$, we get the same structure of flow equations as discussed by Bray and Moore [16]. To do so, we consider eq.(3.21) for an arbitrary scale change; i.e.,we consider a factor b. an arbitrary b would be of order 2^n if eq.(3.21) is to be generalized, and then we consider the limit of $b = 1 + \delta l$. The resulting flow has the structure $\frac{dK}{dl} = -AwK$ where A is an arbitrary const. This holds for $d = 1$. For $d \geq 1$ we note that the contribution will come from the factor of b^{d-1} attached to K, and thus the flow will be of the form

$$\frac{dK}{dl} = K(d - 1 - Aw) \tag{3.26}$$

As for eq.(3.25), it is straightforward to generalize that, since it leads to $\Delta' = 2^d\Delta$ in d dimensions and thus

$$\frac{d\Delta}{dl} = d\,\Delta + (\text{exp. small corrections for } w \ll 1) \tag{3.27}$$

We have in eqs.(3.26) and (3.27) the form by advocated Bray and Moore [16] with the difference that eq.(3.26) is linear in w whereas Bray and Moore argue for a w^2 term. If we introduce an infinitesimal uniform field H, then it is clear that for $T = 0$,$w = 0$

$$\frac{dH}{dl} = H\,d \tag{3.28}$$

For small w there will be small exponential corrections.

In terms of J, H_R and H we can write eqs.(3.26), (3.27) and(3.28) as (omitting exponentially small corrections)

$$\frac{dH_R}{dl} = H_R f_1(w) = H_R.d/2 \tag{3.29}$$

$$\frac{dJ}{dl} = J f_2(w) = J(d - 1 - Aw) \tag{3.30}$$

$$\frac{dH}{dl} = H f_3(w) = H d \tag{3.31}$$

The fixed point is determined from the equation for $w = \frac{H_R}{J}$. Manipulation of eqs.(3.29) and (3.30) leads to

$$\frac{dw}{dl} = w(d - 1 - Aw - d/2) = w(\frac{d-2}{2} - Aw) \tag{3.32}$$

For $d = 2 + \epsilon$, we get

$$\frac{dw}{dl} = w(\frac{\epsilon}{2} - Aw).$$

At the fixed point

$$\frac{dw}{dl}|_{w=w^*} = 0$$

giving $w^* = \frac{\epsilon}{2A}$.

Linearizing around the fixed point gives the exponent ν

$$y_t = 1/\nu = w^*(f_1'(w^*) - f_2'(w^*) = \frac{\epsilon}{2A}(A) = \frac{\epsilon}{2} \tag{3.33}$$

while the y_h and y_J are given by

$$y_h = f_3(w^*) = 2 + \epsilon \tag{3.34}$$

$$y_J = f_1(w^*) = f_2(w^*) = 1 + \epsilon/2 \tag{3.35}$$

to order $O(\epsilon)$.

Using the results given in the table [2.1], the exponents η and $\bar{\eta}$ can also be computed to $O(\epsilon)$, we find

$$\eta = 2 + d + y_J - 2y_h = 1 - \frac{\epsilon}{2}$$

$$\bar{\eta} = 4 + d - 2y_J = 2 - \epsilon$$

Schwartz and Soffer [15] have derived the exact inequality $\bar{\eta} \leq \eta$. To order ϵ this is satisfied as equality. Note that $\bar{\eta} = 2\eta$ implies $y_J = 2 - \eta$. This leads us to the modified hyperscaling law

$$2 - \alpha = d - 2 + \eta \tag{3.36}$$

to $O(\epsilon)$.

Schwartz [14] has argued for exact dimensional reduction for random field magnets where the exponents of d-dimensional RFIM are exactly equal to the exponents of pure system in dimension $d' = d - 2 + \eta$ (d). It is interesting to note that our results are in accord with this rule. Indeed ,$d \rightarrow d - 2 + \eta$ implies that $d' = 1 + \epsilon/2$. The exponents of pure system in dimension $1 + \epsilon/2$ are to $O(\epsilon)$ [52] , $1/\nu = \epsilon/2$, $\eta = 1 - \epsilon/2$, which are in agreement with our results.

Setting $d = 4$ in eq.(3.32) leads to the flow

$$\frac{dw}{dl} = -Aw^2$$

which would give a correlation length $\xi \sim e^{\frac{1}{Aw}}$. Our expression for the correlation length differs from the Binder's [50] estimate of the correlation length $\xi \sim e^{\frac{1}{Aw^2}}$ and that of Grinstein and Ma's [21]$\xi \sim e^{\frac{1}{Aw^{4/3}}}$. Both these results are based on the roughening of the interface due the random field. While Binder's result is based on the discrete version of the Ising model, Grinstein and Ma use the continuum version of the Ising model to get their results. Independent calculation of exponents in $d = 2 + \epsilon$ before us has been done by Bray and Moore [16] and T.Nattermann [22]. These calculations differ from each other and from us in the renormalistion of J. Bray and Moore use first correction to $d - 1$ in eq.(3.26) as $O(w^2)$ so as to get in agreement with Binder's correlation length. Essentially same has been done by Nattermann as regards Grinstein and Ma's results. We believe that discrete as well as continuum version of the Ising model should belong to the same universality class and hence should have the same of critical exponents. Since the calculations based on interface

Table 3.1: List of critical exponents in $d = 2 + \epsilon$

$\beta = 0$	$\nu = \frac{2}{\epsilon}$
$\gamma = \frac{2+\epsilon}{\epsilon}$	$\eta = 1 - \frac{\epsilon}{2}$
$\delta = \infty$	$\bar{\eta} = 2 - \epsilon$

roughening give different values of critical exponents for the two versions, it seems inadequate in explaining the critical behaviour of the random field Ising model.

In Appendix II, we shall derive the R-G flow using the conventional technique where one renormalises the unaveraged hamiltonian and keeps track of the probability distribution of the random field and the spin coupling.

Chapter 4

Random–field Heisenberg Model

For magnets with continuous symmetry ($n \geq 2$) both dimensional reduction and Imry - Ma domain wall arguments yield four as the lower critical dimension. It is thus tempting to speculate that Parisi - Sourla's dimensional reduction holds all the way from six dimension down to four dimension. It was claimed in literature [6] that for $n \geq 2$ to leading order in $\epsilon = d - 4$ the critical behaviour of the random field model is the same as that of pure model in $d = 2 + \epsilon$. Fisher [23], however showed that these calculations are incorrect. The fixed point found by Young [6] in $d = 4 + \epsilon$ was shown to be unstable to perturbation by random anisotropies. These authors start with n - times replicated hamiltonian of the non-linear σ model with random magnetic field. The replicated hamiltonian is then expanded about $\chi = \vec{S}_\alpha.\vec{S}_\beta = 1$ when all the replicas are aligned along the same direction where α and β are replica indices. We believe that it is not consistent to expand about $\chi = \vec{S}_\alpha.\vec{S}_\beta = 1$, when one is interested in the perturbative fixed point in $\frac{H_R}{J}$. Indeed, for $\frac{H_R}{J} = 0$ the replicas are independent and $\langle \vec{S}_\alpha.\vec{S}_\beta \rangle_{\alpha \neq \beta} = 0$. For $\frac{H_R}{J}$ small, $\langle \vec{S}_\alpha.\vec{S}_\beta \rangle_{\alpha \neq \beta}$ is not near one but is near zero. It is perhaps this fact which disables one to find the fixed point in $d = 4 + \epsilon$.

The $\chi = 1$ seems to be the ground state configuration but it is to be noted that the random field related contribution to the partition function from this configuration

is of $O(n^2)$ which has no physical significance. Consequently, our aim will be to avoid the use of replicas in our derivation of the flow equations.

In our analysis below, we avoid replicas and use simple decimation and Migdal - Kadanoff bond moving approximation to study the Heisenberg model near $d = 4 + \epsilon$

The partition function of the Heisenberg model is

$$Z = \int_{|\vec{S}_i|=1\forall i} \prod_{i=1}^{N} d\vec{S}_i \exp\left(\frac{J}{T}\sum_{\langle i,j\rangle} \vec{S}_i.\vec{S}_j + \frac{1}{T}\sum_i \vec{h}_i.\vec{S}_i\right) \tag{4.1}$$

where the random magnetic field has a Gaussian distribution.

$$\langle h_i^\alpha h_j^\beta\rangle = \delta^{\alpha\beta}\delta_{ij}H_R^2 \tag{4.2}$$

$$\langle h_i^\alpha\rangle = H\delta_{1\alpha} \tag{4.3}$$

We shall assume an infinitesimal field in direction 1.

We begin with $d = 1$, the Heisenberg chain in a random magnetic field. The renormalisation procedure would be to trace over alternate spins *i.e.*$S_2^\alpha, S_4^\alpha, \ldots$ etc. at various sites. If we consider the site $2i-1, 2i$, and $2i+1$; the spins at site $2i$ will be removed and partition function is expressed in terms of S_{2i-1}^α and S_{2i+1}^α. After this coarse-graining we replace S_{2i-1}^α by S_i^α. We thus require the relevent bit of Z which we write as

$$\begin{aligned}
Z' &= \int_{S_{2i}} d\vec{S}_{2i} \exp[\frac{J}{T}(\vec{S}_{2i-1}.\vec{S}_{2i} + \vec{S}_{2i}.\vec{S}_{2i+1}) \\
&\quad + \frac{1}{2T}(\vec{h}_{2i-1}.\vec{S}_{2i-1} + 2\vec{h}_{2i}.\vec{S}_{2i} + \vec{h}_{2i+1}.\vec{S}_{2i+1})] \\
&= 2\pi^{n/2}\frac{I_{n/2-1}\left(\frac{J}{T}|\vec{S}_{2i-1} + \vec{S}_{2i+1} + \frac{\vec{h}_{2i}}{J}|\right)}{\left(\frac{J}{T}|\vec{S}_{2i+1} + \vec{S}_{2i-1} + \frac{\vec{h}_{2i}}{J}|\right)^{n/2-1}} \times \\
&\quad \exp[\frac{1}{2T}(\vec{h}_{2i-1}.\vec{S}_{2i-1} + \vec{h}_{2i+1}.\vec{S}_{2i+1})]
\end{aligned} \tag{4.4}$$

Obviously the hamiltonian does not preserve its form after coarse-graining. However, we shall work in the regime where $T \to 0$ and $w = \frac{H_R}{J}$ is very small. In this limit the

average angle between two neighbouring spins $\vec{S}_i$ and $\vec{S}_{i+1}$ is very small and we make the usual spin wave expansion,

$\vec{S}_i.\vec{S}_{i+1} = \cos\theta_{i,i+1} = 1 - \frac{\theta^2_{i,i+1}}{2} +$ (higher order terms in $\theta_{i,i+1}$) .

In the limit of $T \to 0$, we can approximate eq.(4.4) by

$$Z' \approx \exp\left(\frac{J}{T}\left|\vec{S}_{2i-1} + \vec{S}_{2i+1} + \frac{\vec{h}_{2i}}{J}\right| + const\right) \exp\frac{1}{2T}(\vec{h}_{2i-1}.\vec{S}_{2i-1} + \vec{h}_{2i+1}.\vec{S}_{2i+1}) \quad (4.5)$$

We are working in the regime where where we may expand $\frac{J}{T}\left|\vec{S}_{2i-1} + \vec{S}_{2i+1} + \frac{\vec{h}_{2i}}{J}\right|$ as follows (treating $b = \frac{\vec{h}_{2i}.\vec{h}_{2i}}{J^2}$ as the small expansion parameter)

$$\begin{aligned}
\frac{J}{T}\left|\vec{S}_{2i-1} + \vec{S}_{2i+1} + \frac{\vec{h}_{2i}}{J}\right| &= \frac{J}{T}\left(2 + 2\vec{S}_{2i-1}.\vec{S}_{2i+1} + 2\frac{\vec{h}_{2i}}{J}.(\vec{S}_{2i-1} + \vec{S}_{2i+1}) + \frac{\vec{h}_{2i}.\vec{h}_{2i}}{J^2}\right)^{1/2} \\
&= \frac{J}{T}\left(4 - \theta^2_{2i-1,2i+1} + 2\frac{\vec{h}_{2i}}{J}.(\vec{S}_{2i-1} + \vec{S}_{2i+1}) + \frac{\vec{h}_{2i}.\vec{h}_{2i}}{J^2}\right)^{1/2} \\
&= \frac{2J}{T}\left(1 - \frac{\theta^2_{2i-1,2i+1}}{4} + \frac{\vec{h}_{2i}}{2J}.(\vec{S}_{2i-1} + \vec{S}_{2i+1}) + \frac{\vec{h}_{2i}.\vec{h}_{2i}}{4J^2}\right)^{1/2} \\
&= \frac{2J}{T}\left(1 - \frac{1}{8}\theta^2_{2i-1,2i+1}(1 - \frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}) + \frac{1}{4J}\vec{h}_{2i}.(\vec{S}_{2i-1} + \vec{S}_{2i+1})\times\right. \\
&\quad (1 - \frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}) + higher\ orders\ in\ b + terms\ of\ the\ sort\ \theta^{2x}\times \\
&\quad \left.\{\frac{\vec{h}_{2i}}{J}.(\vec{S}_{2i-1} + \vec{S}_{2i+1})^y\},\ and\ [\vec{h}_{2i}.(\vec{S}_{2i-1} + \vec{S}_{2i+1})]^z\right) \qquad (4.6)
\end{aligned}$$

where x, y, and z are integers. New terms in the hamiltonian are generated. We assume that they are irrelevent in the $R-G$ sense at least near the lower the critical dimension. Therefore we may write eq.(4.6) as

$$
\begin{aligned}
\frac{J}{T}\left|\vec{S}_{2i-1}+\vec{S}_{2i+1}+\frac{\vec{h}_{2i}}{J}\right| &= -\frac{J}{4T}\theta^2_{2i-1,2i+1}\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+ \\
&\quad \frac{\vec{h}_{2i}}{2T}.(S_{2i-1}+S_{2i+1})\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+irrelevent\ terms \\
&= -\frac{J}{2T}\vec{S}_{2i-1}.\vec{S}_{2i+1}1\{-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+ \\
&\quad \frac{\vec{h}_{2i}}{2T}.(S_{2i-1}+S_{2i+1})\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+irr.\ terms \qquad (4.7)
\end{aligned}
$$

By substituting eq.(4.7) in eq.(4.5), we end up with

$$
\begin{aligned}
Z'_n &= \exp\Bigg[\frac{J}{2T}\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}\vec{S}_{2i-1}.\vec{S}_{2i+1}+\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}\frac{\vec{h}_{2i}}{2T}.(\vec{S}_{2i-1}+\vec{S}_{2i+1}) \\
&\quad +\frac{1}{2T}(\vec{h}_{2i-1}.\vec{S}_{2i-1}+\vec{h}_{2i+1}.\vec{S}_{2i+1})\Bigg] \qquad (4.8)
\end{aligned}
$$

Similarly we may decimate the spin at site $2i-1$ to arrive at

$$
\begin{aligned}
Z'_n &= \exp\Bigg[\frac{J}{2T}\{1-\frac{\vec{h}_{2i-2}.\vec{h}_{2i-2}}{8J^2}\}\vec{S}_{2i-3}.\vec{S}_{2i-1}+\{1-\frac{\vec{h}_{2i-2}.\vec{h}_{2i-2}}{8J^2}\}\frac{\vec{h}_{2i-2}}{2T}.(\vec{S}_{2i-3}+\vec{S}_{2i-1}) \\
&\quad +\frac{1}{2T}(\vec{h}_{2i-3}.\vec{S}_{2i-3}+\vec{h}_{2i-1}.\vec{S}_{2i-1})\Bigg] \qquad (4.9)
\end{aligned}
$$

After replacing $\vec{S}_{2i-1},\vec{S}_{2i-3}$ and $\vec{S}_{2i+1}$ by $\vec{S}_i,\vec{S}_{i-1}$ and $\vec{S}_{i+1}$ respectively, we get the renormalized J' and h'_i as follows

$$J'_i=\frac{J}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i}h^\alpha_{2i}}{8J^2}\right)+O(b^2) \qquad (4.10)$$

$$h^{\beta'}_i=h^\beta_{2i-1}+\frac{h^\beta_{2i}}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i}h^\alpha_{2i}}{8J^2}\right)+\frac{h^\beta_{2i-2}}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i-2}h^\alpha_{2i-2}}{8J^2}\right)+O(b^2) \qquad (4.11)$$

After taking the average over h^α_i's in eqs. (4.10) and (4.11) by using eqs.(4.2) and (4.3), we end up with

$$J'=\frac{J}{2}\left(1-\frac{nH_R^2}{8J^2}\right)+O(w^4) \qquad (4.12)$$

$$H'=2H\left(1-\frac{(n+2)H_R^2}{16J^2}\right)+O(w^4) \qquad (4.13)$$

$$
\begin{aligned}
\frac{J}{T}\left|\vec{S}_{2i-1}+\vec{S}_{2i+1}+\frac{\vec{h}_{2i}}{J}\right| &= -\frac{J}{4T}\theta^2_{2i-1,2i+1}\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+ \\
&\quad \frac{\vec{h}_{2i}}{2T}.(S_{2i-1}+S_{2i+1})\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+irrelevent\ terms \\
&= -\frac{J}{2T}\vec{S}_{2i-1}.\vec{S}_{2i+1}1\{-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+ \\
&\quad \frac{\vec{h}_{2i}}{2T}.(S_{2i-1}+S_{2i+1})\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}+irr.\ terms \qquad (4.7)
\end{aligned}
$$

By substituting eq.(4.7) in eq.(4.5), we end up with

$$
\begin{aligned}
Z'_n &= \exp\left[\frac{J}{2T}\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}\vec{S}_{2i-1}.\vec{S}_{2i+1}+\{1-\frac{\vec{h}_{2i}.\vec{h}_{2i}}{8J^2}\}\frac{\vec{h}_{2i}}{2T}.(\vec{S}_{2i-1}+\vec{S}_{2i+1})\right. \\
&\quad \left.+\frac{1}{2T}(\vec{h}_{2i-1}.\vec{S}_{2i-1}+\vec{h}_{2i+1}.\vec{S}_{2i+1})\right] \qquad (4.8)
\end{aligned}
$$

Similarly we may decimate the spin at site $2i-1$ to arrive at

$$
\begin{aligned}
Z'_n &= \exp\left[\frac{J}{2T}\{1-\frac{\vec{h}_{2i-2}.\vec{h}_{2i-2}}{8J^2}\}\vec{S}_{2i-3}.\vec{S}_{2i-1}+\{1-\frac{\vec{h}_{2i-2}.\vec{h}_{2i-2}}{8J^2}\}\frac{\vec{h}_{2i-2}}{2T}.(\vec{S}_{2i-3}+\vec{S}_{2i-1})\right. \\
&\quad \left.+\frac{1}{2T}(\vec{h}_{2i-3}.\vec{S}_{2i-3}+\vec{h}_{2i-1}.\vec{S}_{2i-1})\right] \qquad (4.9)
\end{aligned}
$$

After replacing $\vec{S}_{2i-1}, \vec{S}_{2i-3}$ and $\vec{S}_{2i+1}$ by $\vec{S}_i, \vec{S}_{i-1}$ and $\vec{S}_{i+1}$ respectively, we get the renormalized J' and h'_i as follows

$$J'_i = \frac{J}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i}h^\alpha_{2i}}{8J^2}\right)+O(b^2) \qquad (4.10)$$

$$h^{\beta'}_i = h^\beta_{2i-1}+\frac{h^\beta_{2i}}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i}h^\alpha_{2i}}{8J^2}\right)+\frac{h^\beta_{2i-2}}{2}\left(1-\frac{\sum_\alpha h^\alpha_{2i-2}h^\alpha_{2i-2}}{8J^2}\right)+O(b^2) \qquad (4.11)$$

After taking the average over h^α_i's in eqs. (4.10) and (4.11) by using eqs.(4.2) and (4.3), we end up with

$$J' = \frac{J}{2}\left(1-\frac{nH^2_R}{8J^2}\right)+O(w^4) \qquad (4.12)$$

$$H' = 2H\left(1-\frac{(n+2)H^2_R}{16J^2}\right)+O(w^4) \qquad (4.13)$$

$$H_R^{2'} = \frac{3}{2} H_R^2 \left(1 - \frac{(n+2)H_R^2}{12J^2}\right) + O(w^4) \tag{4.14}$$

Unfortunately, renormalisation leads to a new term

$$\lambda' = \langle h_i^{\alpha'} h_{i+1}^{\alpha'} \rangle = \frac{1}{4} H_R^2 \left[1 - \frac{(n+2)H_R^2}{4J^2}\right] + O(w^4)$$

which was zero to start with. The ratio

$$m = \frac{\langle h_i^\alpha h_{i+1}^\alpha \rangle}{\langle h_i^\alpha h_i^\alpha \rangle}$$

increases after each R-G iteration. A proper choice of this ratio would make it invariant under the renormalisation group transformation at the assumed fixed point $w' = w = w*$. This is done by taking

$$\langle h_i^\alpha h_{i+1}^\alpha \rangle = \frac{1}{4} H_R^2 \left(1 - \frac{(n+2)w^2}{8}\right) + O(w^4) \tag{4.15}$$

in the original hamiltonian. After the renormalisation, using eq.(4.10) and (4.11) we get

$$\lambda' = \langle h_i^{\alpha'} h_{i+1}^{\alpha'} \rangle = \frac{1}{2} H_R^2 \left(1 - \frac{(n+2)w^2}{4}\right) + O(w^4) \tag{4.16}$$

$$H_R^{2'} = \langle h_i^{\alpha'} h_i^{\alpha'} \rangle = 2H_R^2 \left(1 - \frac{(n+2)w^2}{8}\right) + O(w^4) \tag{4.17}$$

The ratio

$$m = \frac{\langle h_i^\alpha h_{i+1}^\alpha \rangle}{\langle h_i^\alpha h_i^\alpha \rangle} = \frac{1}{4} \left(1 - \frac{(n+2)w^2}{8}\right) + O(w^4)$$

remains the same after $R-G$ iteration. (ofcourse, w also changes after each $R-G$ iteration, but if we are able to locate a fixed point in w, then $w' = w = w*$). The eq.(4.14) is however modified to eq.(4.17).

These relations hold in one dimension for $T \to 0$ and small w. Using Migdal - Kadanoff bond moving approximation in which we replace $J \to b^{d-1} J$, $H_R^2 \to b^{d-1} H_R^2$ and $H \to b^{d-1} H$; (there is an ambiguity in moving single spin interactions,however,

we can use the same procedure as described in the last chapter) the recursion relations are modified to

$$J' = 2^{d-2}J\left(1 - \frac{nw^2}{2^{d-1}8}\right) + O(w^4) \tag{4.18}$$

$$H' = 2^d H\left(1 - \frac{(n+2)w^2}{2^{d-1}16}\right) + O(w^4) \tag{4.19}$$

$$H'_R = 2^{\frac{d}{2}} H_R\left(1 - \frac{(n+2)w^2}{2^{d-1}16}\right) + O(w^4) \tag{4.20}$$

If w is small the bond moving should be a very good approximation because for most of the cases neighbouring spins will be aligned in the same direction and therefore bond moving costs small energy.

An arbitrary scale change b is given by $b = 2^N$ after N $R-G$ iterations.In the limit $b = 1 + \delta l$ we get the structure of flow equations as discussed by Bray and Moore [16] up to $O(w^2)$

$$\frac{dH_R}{dl} = H_R\left(\frac{d}{2} - \frac{(n+2)Aw^2}{2}\right) + O(w^4) = H_R f_1(w) \tag{4.21}$$

$$\frac{dJ}{dl} = J\left(d - 2 - nAw^2\right) + O(w^4) = J f_2(w) \tag{4.22}$$

$$\frac{dH}{dl} = H\left(d - \frac{(n+2)Aw^2}{2}\right) + O(w^4) = H f_3(w) \tag{4.23}$$

where A is an arbitrary constant. Henceforth we shall drop terms involving higher order corrections in w.

The fixed point R is determined from the equation for $w = \frac{H_R}{J}$. Manipulation of eqs.(4.21) and (4.22) yield the recursion

$$\frac{dw}{dl} = w(f_1(w) - f_2(w)) = w\left(-\frac{d-4}{2} + \frac{(n-2)Aw^2}{2}\right). \tag{4.24}$$

For d = 4, eq.(4.24) leads to the correlation length $\xi \sim \exp\frac{1}{Aw^2}$. Putting $d = 4 + \epsilon$, we get

$$\frac{dw}{dl} = w\left(-\frac{\epsilon}{2} + \frac{(n-2)Aw^2}{2}\right)$$

At the fixed point

$$\frac{dw}{dl}\Big|_{w=w^*} = 0$$

giving

$w^* = {\frac{\epsilon}{(n-2)A}}^{\frac{1}{2}}$.

Linearizing around the fixed point yields the exponent $y_t = \frac{1}{\nu}$,

$$\begin{aligned} y_t &= \frac{1}{\nu} = w^*(f_1'(w^*) - f_2'(w^*)) \\ &= w^*(-(n+2)Aw^* + 2nAw^*) = (n-2)Aw^{*2} = \epsilon \end{aligned} \qquad (4.25)$$

while exponents y_J and y_h are given by

$$y_h = f_3(w^*) = 4 + \epsilon - \frac{(n+2)}{2(n-2)}\epsilon = 4 + \epsilon[1 - \frac{n+2}{2(n-2)}] \qquad (4.26)$$

$$y_J = f_1(w^*) = f_2(w^*) = 2 + \epsilon/2 - \frac{n+2}{2(n-2)} = 2 + \frac{\epsilon}{2}[1 - \frac{n+2}{n-2}] \qquad (4.27)$$

From eqs.(4.25)to (4.27), we can easily find all other critical exponents. The values of critical exponents are listed in the table (4.1).

Schwartz and Soffer [15] have derived the exact inequality $\bar{\eta} \leq 2\eta$. To order ϵ this is satisfied as equality.This would also imply that d-dependent hyperscaling law is modified to '$2 - \alpha = \nu(d - 2 + \eta)$' in presence of the random magnetic field. However there is no exact dimensional reduction,$d \to d - 2 + \eta (= 1 + \frac{n}{n-2}\epsilon)$, for the exponents, since for $d = 1 + \frac{n}{n-2}\epsilon$, the exponents of the pure problem to $O(\epsilon)$ are, $1/\nu = \frac{n}{n-2}\epsilon$ and $\eta = \frac{n}{(n-2)^2}\epsilon$, both being in disagreement with the result given in table (4.1). For $n = 2$ the fixed point remains undetermined. This means that special treatment is required for this case as is the case for the pure model without the random field.

Table 4.1: List of critical exponets in $d = 4 + \epsilon$

$\beta = \frac{n+2}{2(n-2)}$	$\nu = \frac{1}{\epsilon}$
$\gamma = \frac{2(1-\epsilon)}{\epsilon}$	$\eta = \frac{2\epsilon}{n-2}$
$\delta = \frac{2(n-2)}{(n+2)} \frac{2+\epsilon}{\epsilon}$	$\bar{\eta} = \frac{4\epsilon}{n-2}$

Chapter 5

Dynamics

In this chapter we turn our attention to dynamics of the random field Ising model. The critical dynamics of the RFIM is drastically modified from the dynamics of the pure system. Experiments performed on the diluted Ising antiferromagnets (AF) in uniform field – the physical realisation of the RFIM – reveal quite unusual properties. Since the lower critical dimension of the random field Ising model is two , AF should exhibit long range order for low temperatures and small external fields in three dimensions. But when cooled in non-zero fields, AF do not exhibit any long range order down to very low temperatures even for small field strengths. On the other hand, long range order established by cooling in zero fields persists under the application of quite large external fields. The history dependent behaviour implies that it takes very long time for the system to come to equilibrium. Particularly, the critical slowing down is so extreme that it exceeds the largest observation time.

The analysis around the upper critical dimension ($d_u = 6$) indicate that the critical dynamics is of the conventional type with power law divergence [30]. To explain the data in terms of the power law divergence would require very large value of the dynamical exponent [27]–[29]. This value of the dynamical exponent seems unphysical. To account for extremely large times, Villain [19] and Fisher [18] suggested the

activated dynamics where equilibration requires jumps between remote energy wells in phase space.

The dynamics of the model has been the subject of extensive experimental investigation [24] – [27] in an effort to understand whether the critical dynamics is conventional [30]–[31] or of the activated variety [18]–[19]. The initial experiments were not able to resolve the difference between the two forms of dynamics (the span of frequency and equivalently correlation length was not large enough) – activated dynamics could not be differentiated from the conventional dynamics with a very large dynamical scaling exponent(of the order of 10). In the more recent determination of a.c. susceptibility Nash, King and Jaccarino [28] covered a wide range of frequency and could definitely assert that the dynamics was of the activated variety, although the relevent exponent found by them did not agree with the theoretical predictions.In this chapter we apply dynamic scaling and zero temperature fixed point idea to the RFIM dynamics somewhat more carefully than has been done before and find a correction to scaling which serves to reconcile various apparently contradictory features. We show that the exponent θ can be directly determined from the imaginary part of the susceptibility. Further it is demonstrated that this θ, which is larger than that quoted by Nash et al from an analysis of the real part of the susceptibility, is in agreement with there result once the "correction to scaling " has been taken in to account.

The two approaches to critical slowing down have been

1. conventional dynamic scaling [11] where the relaxation time $\tau \sim \xi^z$, ξ being the correlation length. The exponent z is found by constructing a diagrammatic perturbation theory in which the temperature induced fluctuations are completely ignored.

2. dynamic scaling with activated dynamics [19]-[18] being the basis for critical

slowing down, giving $\tau \sim exp(\xi^\theta)$, where θ is the scaling exponent for the temperature field. In this approach the random field is important for producing the metastable configurations ,while temperature is what causes the hopping between metastable states, giving rise to the activated dynamics.

We show that a very straightforward application of the renormalisation group transformation for the relaxation time with the dynamics governed by the zero temperature fixed combines the two pictures given above.We recall briefly the salient features of the zero temperature fixed point. In this case the hamiltonian is no longer scale invariant and under the scale transformation by a factor b , is transformed from H to $b^\theta H$, where θ is an exponent associated with dimensional reduction. Alternatively the temperature can be considered to have transformed and we have the transformation law

$$T' = b^{-\theta} T \tag{5.1}$$

For the conventional fixed point of the critical dynamics without the random field, $\theta = 0$ and the temperature is unchanged.

Now turning to the dynamics we note that under a scale transformation b, the correlation length goes from ξ to ξ/b and the frequency scale goes from ω to $b^{-z}\omega$, where z is a dynamic scaling exponent and hence for the characteristic time τ

$$\tau(\xi, T) = b^z \tau(\frac{\xi}{b}, T') = b^z \tau(\frac{\xi}{b}, T b^{-\theta}) \tag{5.2}$$

Setting $b = \xi$,

$$\tau(\xi, T) = \xi^z \tau(1, T\xi^{-\theta}) = \xi^z f(\frac{T}{\xi^\theta}) \tag{5.3}$$

If $\theta = 0$, it is immediately clear that conventional dynamic scaling is obtained. For $\theta \neq 0$, the functional form of $f(T/\xi^\theta)$ can be crucial. The two factors in eq.(5.3) ξ^z and $f(T/\xi^\theta)$ correspond to the conventional dynamics and activated dynamics respectively.

We need to determine the function $f(T/\xi^\theta)$. To do this we note that this part of τ is clearly produced by the fluctuations induced by the temperature. The random field induced fluctuations in τ are contained in the part ξ^z. In calculating $f(T/\xi^\theta)$, the role of random magnetic field is to produce several metastable states and the temperature provides the noise that leads to hopping between the minima. The effective temperature is T/ξ^θ and is very small for large ξ. This dynamics can be taken to be governed at temperature T by a Fokker Planck equation of the form

$$\frac{\partial P}{\partial t} = \vec{\nabla}.(P\vec{\nabla}V) + kT\vec{\nabla}^2 P \tag{5.4}$$

where P is the probability of a particular field configuration at a time t and the scalar field $\phi(x)$ is the coarse-grained order parameter for the RFIM. The potential $V(\phi)$ is the one determining the deterministic drive in the Langevin Equation for $\phi(x)$. The potential $V(\phi)$ is characterised by the existence of several metastable minima. In zero dimensions $\phi(x)$ is an algebraic variable, the Fokker Planck equation is one dimensional and it is a well known result that the largest relaxation time in the system has the structure $exp(\Delta V/T)$ where ΔV is the typical height of the potential barrier. In non–zero spatial dimension, the Fokker Planck equation is infinite dimensional corresponding to infinite number of Fourier components of $\phi(x)$. The relaxation rate is still of the form $\exp(\Delta V/T)$, provided tunnelling path exists [32]–[33] connecting the different metastable minima. The typical potential barrier is of order unity at these low temperatures and hence at temperatures of T/ξ^θ

$$f(\frac{T}{\xi^\theta}) \sim exp(\frac{a\xi^\theta}{kT}) \tag{5.5}$$

leading to

$$\tau = \tau_0(\frac{\xi}{\xi_0})^z \exp(\tau_1(\frac{\xi}{\xi_0})^\theta) \tag{5.6}$$

where τ_0 and τ_1 are system dependent constants. If the metastable minima are ignored, which is equivalent to taking $\theta = 0$ (temperature does not scale), the relaxation rate would be $\tau = \tau_0\xi^z$, with z obtained from a perturbative expansion around

$D = 6$, leading to

$$z = 2 + 2\eta \tag{5.7}$$

Our principle result is that given in eq.(5.6). We first point out that if τ is taken to be dominated purely by activated dynamics, then eq.(5.6) implies that

$$\tau = \tau_0 \exp(\tau_1 (\frac{\xi}{\xi_0})^{\theta_{eff}}) \tag{5.8}$$

where

$$\theta_{eff} = \theta - \frac{z}{\tau_1} \frac{(ln \frac{\xi}{\xi_0}^{\theta} - 1)}{\frac{\xi}{\xi_0}^{\theta} + \frac{z}{\tau_1} ln \frac{\xi}{\xi_0}} \tag{5.9}$$

For reasonable correlation lengths it is clear from eq.(5.9) that $\theta_{eff} < \theta$. It should be noted that the coefficient z/τ_1 of the correction term in eq.(5.9) is not necessarily a small term and hence the correction to the exponent θ can continue to be effective well into the critical regime. In the experiments of King et al [26], the largeness of z/τ_1 leads to the conclusion that the critical behaviour may be governed by activated dynamics or by conventional dynamics with a fairly large value of z. The experiments of Nash et al [28] are carried out in a more asymptotic range and hence the critical behaviour of activated dynamics is seen. However, the exponent θ is lowered from its zero temperature fixed point value.

If the data were analysed in terms of conventional dynamic scaling on the other hand, eq.(5.6)would imply

$$\tau = \tau_0 (\frac{\xi}{\xi_0})^{z_{eff}} \tag{5.10}$$

where

$$z_{eff} = z + \theta \tau_1 (\frac{\xi}{\xi_0})^{\theta} \tag{5.11}$$

If τ_1 is a number considerably smaller than unity, this would be reasonable for small ξ, but would yield large values of z_{eff} for large correlation lengths. The typical z obtained by King et al [26] was around 14. We believe that the same situation holds for the experiments of Geshwind et al [29].

We now address the question of applying dynamical scaling when the relaxation time is given by eq.(5.6). To set up dynamic scaling we need to express ξ^θ in terms of the relaxation time τ. Noting that $\exp \xi^\theta$ will dominate ξ^z numerically over the critical region, we immediately find

$$\xi^\theta \sim ln(\frac{\tau}{(ln\tau)^{\frac{z}{\theta}}}) \tag{5.12}$$

For the temperature and frequency dependent susceptibility χ (or any response function that is measured), the critical behaviour will be given by

$$\chi(\omega,\xi) \sim \xi^x G\left(\frac{ln\left(\omega(|ln\omega|)^{\frac{z}{\theta}}\right)}{\xi^\theta}\right) \tag{5.13}$$

The exponent x determines the zero frequency, that is the thermodynamic behaviour. If we now pass to the extreme non-thermodynamic limit i. e. $\xi \to \infty$ and the frequency alone determines the response, we have for a finite response

$$\chi(\omega, T = T_c) \sim \left(ln\left|\omega(|ln\omega|)^{\frac{z}{\theta}}\right|\right)^{\frac{x}{\theta}} \tag{5.14}$$

For $x \to 0$ (which is what happens for this response)

$$\chi(\omega, T = T_c) \sim ln\left(ln\left|\omega(|ln\omega|)^{\frac{z}{\theta}}\right|\right)$$

A frequency dependent response at $T = T_c$ will be a complex quantity with the real and imaginary part following the Kramers–Kronig relations. This is due to causality, which can be ensured by replacing ω by $-i\omega$ in eq.(5.14). Hence at $T = T_c$,

$$\chi_1 + i\chi_2 = \chi_0\left(ln(-i\omega[ln(-i\omega)]^{\frac{z}{\theta}})\right)^{\frac{x}{\theta}} \tag{5.15}$$

If we approximate $z/\theta \sim 1$ and consider a frequency range where $|ln\omega|$ is considerably larger than $\pi/2$, then the imaginary part χ_2 of χ has the form (for $x \to 0$)

$$\chi_2 \sim \frac{1}{ln\,|\omega ln\omega|} \tag{5.16}$$

Clearly an order of magnitude smaller than the real part χ_1, which scales as $ln\left(|ln\,\omega(|ln\omega|)\xi|\right)$. The prediction of eq.(5.16) should be easily verifiable experimentally.

We will now demonstrate that in the hydrodynamic regime (small ξ), the rise of χ_2 with increasing ξ can be quantitatively obtained from dynamic scaling considerations. The hydrodynamic regime in the present case is obtained for $\xi^\theta / ln\,|\omega ln\omega| \ll 1$. Writing the scaling form of χ as in eq.(5.13) with ω replaced by $-i\omega$, we find that the leading term in χ in the hydrodynamic regime is

$$\chi \sim \xi^x \frac{\xi^\theta}{ln(-i\omega ln\omega)} = \xi^x \frac{\xi^\theta}{ln(\omega ln\omega) - i\frac{\pi}{2}} = \frac{\xi^{\theta+x}}{ln(\omega ln\omega)} + \frac{i\pi}{2}\frac{\xi^{x+\theta}}{(ln|\omega ln\omega|)^2} \tag{5.17}$$

leading to

$$\chi_2 \sim \xi^{x+\theta}\frac{1}{(ln\,|\omega ln\omega|)^2} \tag{5.18}$$

For $x \to 0$, we have the prediction

$$\chi_2 \sim \xi^\theta \frac{1}{(ln\,|\omega ln\omega|)^2} \tag{5.19}$$

At a fixed frequency, the imaginary part of the susceptibility gives a direct determination of θ and using the data of Nash et al., we find $\theta = 1.25$.

We now show that this θ and that obtained from eq.(5.9) are consistent. To do so we note that the range of ω_0/ω for the experiment of Nash et al is 10^8 to 10^3. The middle of the range is at about $10^{5.5}$ which leads to $(\xi/\xi_0)^\theta$ of ~ 10 from eq.(5.12). The earlier analysis of King et al, which was in terms of a power law yielded an exponent of about 14. From eq(5.11), this corresponds to $\tau_1 \approx 0.8$. If we now use this value of τ_1 in eq.(5.9) , we come up with $\theta - \theta_{eff} \approx 0.20$. With the θ_{eff} as found by Nash et al as 1.05 ± 0.16, we find $\theta \approx 1.25$ in agreement with that found from the imaginary part of the susceptibility.

We end by providing the results for the case where the wavenumber is finite. The finite frequency, finite wavenumber response function $\chi(\xi, \omega, k)$ should be amenable

to neutron scattering experiments. The dynamic scaling requirement would give the form

$$\chi(\xi,\omega,k) = \xi^{x} F\left(\frac{ln(w\,|(ln\omega|^{\frac{x}{\theta}})}{\xi^{\theta} f(k\xi)}, k\xi\right) \tag{5.20}$$

Clearly for $k\xi \to 0$, $f(k\xi)$ must tend to unity. The more interesting limit is clearly as $k\xi \to \infty$ i. e. approaching the critical point ($\xi = \infty$) at finite k and ω, In this limit clearly

$$\chi(\omega,k) = k^{-x} H\left(\frac{ln(w|ln\omega|^{\frac{x}{\theta}})}{k^{-\theta}}\right) \tag{5.21}$$

which implies that the k dependent relaxation rate at $\xi = \infty$ goes as

$$\Omega(k) \sim k^{z} \exp(-k^{-\theta})$$

This relation can be probed by neutron scattering at finite k and ω as the critical temperature is approached.

Chapter 6

One Dimensional Models

6.1 Ginzburg-Landau Model: Equivalent Quantum Mechanics

While the random field Ising model has seen a lot of activity, the low dimensional Ginzburg-Landau model has not attracted much attention. In low dimensions (i.e. $D = 1$ or 0), there is no phase transition (not even at $T = 0$ as there is in the absence of the random field) and any calculation of a thermodynamic quantity, e.g. the free energy, must produce a result which is valid over a large temperature range. For a one dimensional Ginzburg-Landau model without the random field, this has been done by Ferrell and Scalpino [34] and by Balian and Toulouse [35] who showed that the problem of calculating the free energy of the model could be reduced to the problem of calculating the ground state energy of a quantum mechanical problem. For the zero dimensional Ginzburg-Landau model in a random external field , Bray and McKane [36] used the replica trick to evaluate the partition function. In this section, we treat the the one dimensional Ginzburg-Landau model in a random external field by a combination of the two techniques. We use the replica trick and

then analytically continue to imaginary values of the spatial coordinate to set up the equivalent quantum problem. The ground state energy now leads to the free energy.

The free energy functional for the scalar Ginzburg–Landau in an external field is

$$F = \int_0^L dx \left[\frac{1}{2} r_0 M^2(x) + \frac{1}{2} \left[\frac{\partial M(x)}{\partial x} \right]^2 + \frac{u}{4} M^4(x) - M(x) h(x) \right] \tag{6.1}$$

The random field $h(x)$ has the probability distribution

$$P(h) = \exp\left[-\int \frac{h^2(x)dx}{4\Delta} \right] \tag{6.2}$$

and we are interested in the partition function

$$Z[h(x)] = \int D[M] \exp -\frac{1}{T} \left[-\int_0^L F(x)dx \right] \tag{6.3}$$

A change of scale $\phi = M\frac{1}{T^{1/2}}$ on the field variable yields

$$Z[h(x)] = T^{\frac{L}{2}} \int D[\phi] \exp\left[-\int_0^L L[\phi]dx \right] \tag{6.4}$$

with

$$L[\phi] = \frac{1}{2} r_0 \phi^2(x) + \frac{1}{2} \left[\frac{\partial \phi(x)}{\partial x} \right]^2 + \frac{uT}{4} \phi^4(x) - \frac{1}{T^{1/2}} \phi(x) h(x) \tag{6.5}$$

The differential element $D[\phi]$ is defined as usual as a limit of a product over discrete values of x and a is some lattice spacing. From the partition function the thermodynamic free energy can be obtained by averaging over h and is given as

$$G = \langle lnZ \rangle_h \tag{6.6}$$

Differentiation now gives the specific heat. The difficulty of calculating $Z[h(x)]$ and then performing the h average is circumvented by the replica trick, whereby we study the n-times replicated system

$$Z^n = T^{\frac{nL}{2}} \int \prod_{\alpha=1}^{n} D[\phi_\alpha] \exp[-L] \tag{6.7}$$

where

$$L = \frac{1}{2}r_0\sum_\alpha \phi^2(x) + \frac{1}{2}\sum_\alpha\left[\frac{\partial\phi(x)}{\partial x}\right]^2 + \frac{uT}{4}\sum_\alpha\phi^4(x) - \frac{1}{T^{1/2}}\sum_\alpha\phi(x)h(x) \qquad (6.8)$$

The replica index α runs from 1 to n. Now the h average can be performed to yield

$$\begin{aligned}\langle Z^n\rangle &= \int\prod_{\alpha=1}^{n} D[\phi_\alpha]D[h(x)]\exp\left[-\int_0^L dx\left[\frac{1}{2}r_0\sum_\alpha\phi_\alpha^2(x)+\right.\right.\\ &\left.\left.\frac{1}{2}\sum_\alpha\left[\frac{\partial\phi_\alpha(x)}{\partial x}\right]^2 + \frac{uT}{4}\sum_\alpha\phi_\alpha^4(x) - \sum_\alpha\phi_\alpha(x)h(x)\right]\right]\\ &\exp\left[-\int\frac{h^2(x)dx}{4\Delta}\right]\\ &= \int\prod_{\alpha=1}^{n}\exp\left[-\int_0^L dx L_{eff}\right] \qquad (6.9)\end{aligned}$$

where

$$L_{eff} = \frac{1}{2}r_0\sum_\alpha\phi_\alpha^2(x) + \frac{1}{2}\sum_\alpha\left[\frac{\partial\phi_\alpha(x)}{\partial x}\right]^2 + \frac{uT}{4}\sum_\alpha\phi_\alpha^4(x) - \frac{\Delta}{T}\left[\sum_\alpha\phi_\alpha\right]^2. \qquad (6.10)$$

If we analytically continue the coordinate to imaginary values i. e. consider $x \to it$, then L_{eff} becomes the lagrangian for a quantum mechanical problem with

$$\begin{aligned}L &= \frac{1}{2}\left[\sum_\alpha\left[\frac{\partial\phi_\alpha}{\partial t}\right]^2 - \left[\frac{1}{2}r_0\sum_\alpha\phi_\alpha^2(x) + \frac{u}{4}\sum_\alpha\phi_\alpha^4(x) - \Delta\left[\sum_\alpha\phi_\alpha\right]^2\right]\right].\\ &= \frac{1}{2}\left[\sum_\alpha\left[\frac{\partial\phi_\alpha}{\partial t}\right]^2 - V(\phi_\alpha)\right] \qquad (6.11)\end{aligned}$$

We can consider the n variables ϕ_α as the coordinates of a particle moving in a n-dimensional potential well $V(\phi)$. The quantum hamiltonian associated with the lagrangian is given by

$$H = -\frac{1}{2}\sum_\alpha\frac{\partial^2}{\partial\phi_\alpha^2} + \left[\frac{1}{2}r_0\sum_\alpha\phi_\alpha^2(x) + \frac{uT}{4}\sum_\alpha\phi_\alpha^4(x) - \frac{\Delta}{T}\left[\sum_\alpha\phi_\alpha\right]^2\right] \qquad (6.12)$$

Denoting the eigenvalues of H by ϵ_α, the partition function $\langle Z^n\rangle$ at temperature T in the limit $L \to \infty$ is given by

$$\langle Z^n\rangle = e^{-\epsilon_0(n)L}T^{\frac{nL}{2a}} \qquad (6.13)$$

where $\epsilon_0(n)$ is the ground state energy of the hamiltonian.

One can easily calculate the thermodynamic free energy density as

$$F = -\frac{T}{L}\langle lnZ\rangle = T\lim_{n\to 0}\frac{d\epsilon_0(n)}{dn} - \frac{T}{2a}\ln T \tag{6.14}$$

We now proceed to find the ground state energy of the hamiltonian of eq.(6.12) by writing it as

$$H = -\frac{1}{2}\sum_\alpha \frac{\partial^2}{\partial\phi_\alpha^2} + \left[\frac{1}{2}r_0\sum_\alpha \phi_\alpha^2(x) + \frac{uT}{4}\sum_\alpha \phi_\alpha^4(x) - \frac{\Delta}{T}\left[\sum_\alpha \phi_\alpha\right]^2\right] \tag{6.15}$$

Our strategy is to split H as

$$H = \sum_\alpha H_{0\alpha} + H_I \tag{6.16}$$

where

$$H_{0\alpha} = -\frac{1}{2}\frac{\partial^2}{\partial\phi_\alpha^2} + \frac{1}{2}r_0\phi_\alpha^2(x) + \frac{uT}{4}\phi_\alpha^4 \tag{6.17}$$

and

$$H_I = -\Delta\sum_{\alpha\beta}\phi_\alpha\phi_\beta \tag{6.18}$$

The hamiltonian $H_{0\alpha}$ will now be replaced by equivalent harmonic oscillator hamiltonian $\bar{H}_{0\alpha}$, for the ground state, such that

$$\bar{H}_{0\alpha} = -\frac{1}{2}\frac{\partial^2}{\partial\phi_\alpha^2} + \frac{1}{2}\omega_\alpha^2(\phi_\alpha - \phi_0)^2 - \frac{1}{2}\omega^2\phi_0^2, \tag{6.19}$$

where ϕ_0 is present only if $r_0 < 0$

One can now diagonalize $\sum_\alpha \bar{H}_{0\alpha} + H_I$ to find the approximate ground state energy for H. Clearly, the task would be to find as accurate an answer for w_α as possible. Accordingly we have to worry about two distinct regimes $(i) r_0 > 0$ and $(ii) r_0 < 0$. In the case when $r_0 > 0$ we write

$$\begin{aligned} H_{0\alpha} &= -\frac{1}{2}\frac{\partial^2}{\partial\phi_\alpha^2} + \frac{1}{2}\omega_\alpha^2\phi_\alpha^2 - \frac{u}{4}(\phi_\alpha^4 - 3\langle\phi_\alpha^2\rangle\phi_\alpha^2) \\ &= H_{0\alpha} - \frac{uT}{4}(\phi_\alpha^4 - 3\langle\phi_\alpha^2\rangle\phi_\alpha^2) \end{aligned} \tag{6.20}$$

where

$$\omega_\alpha^2 = r_0 + \frac{3uT}{2}\langle\phi_\alpha^2\rangle, \tag{6.21}$$

and$\langle\phi_\alpha^2\rangle$ is the expectation value of ϕ_α^2 in the ground state energy of the hamiltonian $\bar{H}_{0\alpha}$. The self consistency is assured by treating the part $\frac{u}{4}(\phi_\alpha^4 - 3\langle\phi_\alpha^2\rangle\phi_\alpha^2)$ as a perturbation and requiring that the first contribution to the ground state energy vanish. This leads to the condition

$$\langle\phi_\alpha^4\rangle = 3\langle\phi_\alpha^2\rangle \tag{6.22}$$

which is automatically satisfied by our chosen hamiltonian $\bar{H}_{0\alpha}$, and hence from eq.(6.21), ω_α^2 is given by

$$\omega_\alpha^2 = r_0 + \frac{3uT}{2\omega_\alpha} \tag{6.23}$$

Clearly, ω_α^2 is independent of the index α. For $r > 0$, there is only one positive root of eq.(6.23) in ω_α. We immediately see that eq.(6.15)can now be diagonalized as

$$H = -\frac{1}{2}\sum_\alpha \frac{\partial^2}{\partial\phi_\alpha^2} + \frac{1}{2}\sum_\alpha \bar{\omega}_\alpha^2\phi_\alpha^2 \tag{6.24}$$

where $\bar{\omega}_\alpha^2 = \omega^2 - 2n\frac{\Delta}{T}$ for $\alpha = 1$ (say) and equals ω^2 for the remaining $(n-1)$ values of α. We can now immediately write down the ground state energy as

$$\epsilon_0(n) = \frac{1}{2}[\omega^2 - 2n\frac{\Delta}{T}] + \frac{1}{2}(n-1)\omega \tag{6.25}$$

where ω is the solution of eq.(6.23). The free energy follows from eqs.(6.14)and (6.25) as

$$F = \frac{1}{2}T[\omega - \frac{\Delta}{T\omega}] - \frac{T}{2a}lnT \tag{6.26}$$

$r_0 < 0$: In this case there are several extrema and $\psi_\alpha = 0$ corresponds now to a maximum. To find the minima we differentiate $V(\phi)$ with respect to ϕ_α.

$$V(\phi) = \left[\frac{1}{2}r_0\sum_\alpha \phi_\alpha^2(x) + \frac{uT}{4}\sum_\alpha \phi_\alpha^4(x) - \frac{\Delta}{T}\left[\sum_\alpha \phi_\alpha\right]^2\right] \tag{6.27}$$

$$V'(\phi) = r_0\phi_\alpha + \frac{uT}{4}\phi_\alpha^3 - 2\frac{\Delta}{T}\sum_{\beta=0}^{n}\phi_\beta \tag{6.28}$$

Setting $V'(\phi) = 0$ and considering the replica symmetric solution we get

$$V'(\phi*) = r_0\phi * + \frac{uT}{4}\phi*^3 - 2n\frac{\Delta}{T}\phi*$$

which implies

$$\phi* = \pm\sqrt{\frac{-r_0 + 2n\Delta/T}{uT}}.$$

Using the shifted variable $\psi = \phi_\alpha - \phi*$, we may write the hamiltonian H as

$$\begin{aligned} H &= -\frac{1}{2}\sum_\alpha \frac{\partial^2}{\partial\psi_\alpha^2} + \frac{1}{2}(-2r_0 + \frac{6n\Delta}{T}\sum_\alpha \psi_\alpha^2 + \sqrt{uT(-r_0 + 2n\frac{\Delta}{T})}\sum_\alpha \psi_\alpha^3 + \\ &\quad \frac{uT}{4}\sum_\alpha \psi_\alpha^4 - \frac{\Delta}{T}\sum_{\alpha\beta}\psi_\alpha\psi_\beta - \frac{nr_0^2}{4uT} + n^2 \quad \text{(constant)} \\ &= \sum_\alpha H_{0\alpha} + H_I \end{aligned} \tag{6.29}$$

where

$$H_{0\alpha} = -\frac{1}{2}\frac{\partial^2}{\partial\psi_\alpha^2} + \frac{1}{2}(-2r_0)\psi_\alpha^2 + \sqrt{uT(-r_0)}\psi_\alpha^3 + \frac{uT}{4}\psi_\alpha^4 - \frac{r_0^2}{4uT} \tag{6.30}$$

and

$$H_I = -\frac{\Delta}{T}\sum_{\alpha\beta}\psi_\alpha\psi_\beta \tag{6.31}$$

We have dropped those terms in $H_{0\alpha}$ which give $O(n^2)$ contribution to the ground state energy. By re–adjusting the terms in $H_{0\alpha}$, we may write it as

$$\begin{aligned} H_{0\alpha} &= -\frac{1}{2}\frac{\partial^2}{\partial\psi_\alpha^2} + \frac{1}{2}(2|r_0| + \frac{uT}{2}C_2\langle\psi_\alpha^2\rangle)\psi_{\alpha^2} + C_1\sqrt{uT|r_0|}\langle\psi_\alpha^2\rangle\psi_\alpha \\ &\quad + \sqrt{uT|r_0|}(\psi_\alpha^3 - C_1\langle\psi_\alpha^2\rangle\psi_\alpha) + \frac{uT}{4}(\psi_\alpha^4 - C_2\psi_\alpha^2\langle\psi_\alpha^2\rangle) - \frac{r_0^2}{4uT} \end{aligned} \tag{6.32}$$

We now replace $H_{0\alpha}$ by the equivalent harmonic oscillator $\bar{H}_{0\alpha}$ for the ground state. We identify

$$\bar{H}_{0\alpha} = -\frac{1}{2}\frac{\partial^2}{\partial\psi_\alpha^2} + \frac{1}{2}(2|r_0| + \frac{uT}{2}C_2\langle\psi_\alpha^2\rangle)\psi_\alpha^2 + C_1\sqrt{uT|r_0|}\langle\psi_\alpha^2\rangle\psi_\alpha - \frac{r_0^2}{4uT} \tag{6.33}$$

and calculate all the average values with respect to the ground state wave function of the hamiltonian. Accordingly

$$\langle\psi_\alpha\rangle = -C_1\frac{\sqrt{u|r_0|T}}{\omega_\alpha^2}\langle\psi_\alpha^2\rangle \tag{6.34}$$

$$\langle\psi_\alpha^2\rangle = \frac{1}{\omega_\alpha}(1+\delta) \tag{6.35}$$

$$\langle\psi_\alpha^3\rangle = -3C_1\frac{\sqrt{u|r_0|T}}{\omega_\alpha^3}(1+\delta/3) \tag{6.36}$$

$$\langle\psi_\alpha^4\rangle = \frac{3}{\omega_\alpha^2}(1+2\delta+\delta^2/3) \tag{6.37}$$

where

$$\delta = C_1^2 b^2\frac{\langle\psi_\alpha^2\rangle^2}{\omega_\alpha^3} \tag{6.38}$$

and

$$\omega_\alpha^2 = 2|r_0| + \frac{uT}{2}C_2\langle\psi_\alpha^2\rangle \tag{6.39}$$

The correction factor δ is small in all regimes of uT and hence the consistency condition of

$$\langle\psi_\alpha^4\rangle = C_2\langle\psi_\alpha^2\rangle \tag{6.40}$$

and

$$\langle\psi_\alpha^3\rangle = \langle\psi_\alpha^2\rangle\langle\psi_\alpha\rangle \tag{6.41}$$

sets $C_1 = C_2 \approx 3$. We determine $\langle\psi_\alpha^2\rangle = \langle\psi^2\rangle$ (since $\langle\psi_\alpha^2\rangle$ is independent of α) from eqs.(6.35), (6.38) and (6.39) as

$$\psi^2 = \frac{1}{2[|r_0| + (3uT/2)\langle\psi^2\rangle]^2}\left[1+\frac{9uT|r_0|\langle\psi^2\rangle)^2}{[2|r_0| + (3uT/2)(\langle\psi^2\rangle)]^{3/2}}\right] \tag{6.42}$$

and as a byproduct we write down the approximate ground state energy of $H_{0\alpha}$ as

$$\epsilon_0 = \frac{1}{2}\left[2|r_0| + \frac{3uT}{2}\langle\psi^2\right]^{1/2} - \frac{9}{2}\frac{uT|r_0|(\langle\psi^2\rangle)^2}{2r_0 + 3/2uT\langle\psi^2\rangle} - \frac{r_0^2}{4uT}. \tag{6.43}$$

by using eqs.(6.33) and (6.34). This will correspond to the case when there is no random magnetic field.

An immediate question is, how good is this approximation ? For $uT = 1$, we can compare with the numerical results of Ferrell and Scalpino [34]. The agreement is excellent. In particular, the position of zero of $\epsilon(r_0)$ (the most sensitive test of accuracy of the approximation) is reproduced for within 5%.

Having determined the equivalent harmonic oscillator, we now turn to tackle H_I. We are faced with the problem of finding the ground state energy of

$$\bar{H} = \sum_{\alpha} \bar{H}_{0\alpha} - \frac{\Delta}{T} \sum_{\alpha\beta} \psi_\alpha \psi_\beta \tag{6.44}$$

This can be achieved by the orthogonal transformation

$$\psi = O\tau$$

where O is the orthogonal matrix

$$O = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{[n(n-1)]^{1/2}} & \frac{1}{[n(n-1)]^{1/2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{n}} & \frac{1}{[n(n-1)]^{1/2}} & \frac{1}{[n(n-1)]^{1/2}} & \cdots & -\frac{1}{\sqrt{2}} \\ . & . & . & \cdots & 0 \\ . & . & . & \cdots & 0 \\ . & . & . & \cdots & 0 \\ . & . & . & \cdots & 0 \\ . & . & -\frac{(n-2)}{[(n-1)(n-2)]^{1/2}} & \cdots & 0 \\ \frac{1}{\sqrt{n}} & -\frac{(n-1)}{[n(n-1)]^{1/2}} & 0 & \cdots & 0 \end{bmatrix} \tag{6.45}$$

With this transformation

$$\bar{H} = -\frac{1}{2}\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial \tau_\alpha^2} + \frac{1}{2}[\omega^2 - 2n\frac{\Delta}{T}]\tau_1^2 + \frac{1}{2}\sum_{\beta=2}^{n} \omega \tau_\beta^2 + 3\sqrt{nuT|r_0|}\tau_1 - \frac{nr_0^2}{4uT} \tag{6.46}$$

The ground state energy is now easily seen to be

$$\epsilon_0(n) = \frac{1}{2}(n-1)\omega + \frac{1}{2}(\omega^2 - 2n\frac{\Delta}{T})^{1/2} - 9n\frac{uT|r_0|\langle\psi^2\rangle^2}{\omega^2 - 2n\Delta} - \frac{nr_0^2}{4uT} \tag{6.47}$$

from which free energy follows as

$$F = T\frac{d\epsilon}{dn}|_{n=0} - \frac{T}{2a}lnT \tag{6.48}$$

$$= \frac{T\omega}{2} - \frac{\Delta}{2\omega} + \frac{9uT|r_0|\langle\psi^2\rangle^2}{\omega^2}T - \frac{r_0^2}{4u} - \frac{T}{2a}lnT \tag{6.49}$$

where

$$\omega^2 = 2|r_0| + \frac{3uT}{2}\langle\psi_\alpha^2\rangle \tag{6.50}$$

and $\langle\psi^2\rangle$ is given by eq.(6.42).

Discussion: We note that as compared to the no external field case of Ferrell and Scalpino, free energy is decreased for all values of r_0. For $r_0 > 0$ the decrement decreases as r_0 is increased. If we define

$$\Delta F(r_0) = F(r_0) - F(r_0)\,|_{nofield}$$

then $\Delta F(r_0)$ is negative definite, goes to zero as $|r_0| \to \infty$ and has a peak at negative value of r_0. It is instructive to define the entropy function $S(r_0) = -\frac{dF(r_0)}{dr_0}$, from which we see that because of the effect of random field , the entropy is increased for all values of r_0. We conclude with a discussion of specific heat which is given by $C = -\frac{d^2F}{dr_0^2}$. In the absence of random field, there is a peak in the specific heat at a negative value of r_0. In the presence of random field the peak still exists. In fact, the peak is more pronounced as Δ-term makes a positive contribution. For positive values of r_0 and $r_0 \gg 1$, the specific heat goes to zero as it must reproduce the $\Delta = 0$ limit. The behaviour of the specific heat in the one dimensional random field should be accessible experimentally and therein lies the utility of the above results.

6.2 Random Field Ising Model

The random field Ising model in one dimension has attracted considerable attention in recent years [46] - [49]. Though there is no phase transition in one dimension, the presence of random field introduces new features which are absent in the pure problem. In this section we will estimate the free energy of RFIM in a nonperterbative manner by using the replica trick.

The hamiltonian of the random field Ising model in one dimension is

$$H = -J\sum_i S_i S_{i+1} - \sum_i h_i S_i \tag{6.51}$$

where h_i is random with Gaussian probability distribution

$$\langle h_i h_i \rangle = \delta_{ij} H_R^2 \tag{6.52}$$

. We are interested in obtaining the thermodynamic free enery

$$\langle F \rangle_h = \langle \ln Z \rangle_h \tag{6.53}$$

As mentioned in the previous section, this may be obtained by using the replica trick, whereby we study the n-times replicated system. The average partition function of the n-times replicated system is given by

$$\begin{aligned} \langle Z^n \rangle &= \sum_{\{S_i^\alpha\}} \exp\left(\frac{J}{T}\sum_{i\alpha} S_i^\alpha S_{i+1}^\alpha + \frac{H_R^2}{T^2}\sum_{i\alpha\beta} S_i^\alpha S_i^\beta\right) \\ &= \sum_{\{S_i^\alpha\}} \exp\left(K\sum_{i\alpha} S_i^\alpha S_{i+1}^\alpha + \Delta\sum_{i\alpha\beta} S_i^\alpha S_i^\beta\right) \end{aligned} \tag{6.54}$$

where $K = \frac{J}{T}$ and $\Delta = \frac{H_R^2}{J^2}$.

Using the familiar transfer matrix method, the evaluation of the partition function $\langle Z^n \rangle$ of one dimensional chain can be reduced to the computation of the largest eigenvalue of the $2^n \times 2^n$ matrix $T(s, s')$. The transfer matrix $T(s, s')$ has the form

$$T(s, s') = \exp[\frac{\Delta}{2}\sum_{\alpha,\beta}^{n} S^\alpha S^\beta] \exp[K\sum_{\alpha=1}^{n} S^\alpha S'^\alpha] \exp[\frac{\Delta}{2}\sum_{\alpha,\beta}^{n} S'^\alpha S'^\beta] \tag{6.55}$$

where $S = (S^1, S^2, ..., S^n)$ and $S' = (S'^1, S'^2, ..., S'^n)$

In the thermodynamic limit, the free energy is given by

$$F = T\frac{d\lambda(n)}{dn}|_{n=0} \tag{6.56}$$

The procedure relies on the fact that the form of $\lambda(n)$ would permit analytical continuation in n. Since we are unable to diagonalise the matrix $T(s, s')$ for any arbitrary n, we calculate the approximate largest eigenvalue by using a method which is equivalent to using trial wave function in quantum mechanics.

We consider the trial column matrix

$$a = \begin{pmatrix} e^{n^2\Delta/2} \\ e^{(n-2)^2\Delta/2} \\ \vdots \\ ({}^nC_1\text{elements}) \\ e^{(n-4)^2\Delta/2} \\ \vdots \\ ({}^nC_2\text{elements}) \\ e^{(n-6)^2\Delta/2} \\ \vdots \end{pmatrix} \tag{6.57}$$

This choice of elements of the column matrix is motivated by the fact that it is the eigenvector of $T(s, s')$ which corresponds to the largest eigenvalue $\lambda(n)$ in two extreme limits, namely when

1. $K =$ any arbitrary value and $\Delta = 0$
2. $K = 0$ and $\Delta =$ any arbitrary value.

The largest eigenvalue may now be computed as

$$\lambda(n) \approx \frac{\langle a|T|a\rangle}{\langle a|a\rangle} \quad (\textit{in the Dirac's bra} - \textit{ket notation}) \tag{6.58}$$

We have numerically checked this for $n = 2$ and $n = 3$ when it is possible to diagonalise the transfer matrix T. For all T and Δ that were numerically checked, we found

$$\frac{\lambda(2) - \frac{\langle a|T|a\rangle}{\langle a|a\rangle}}{\lambda(2)} \leq 0.18.$$

By taking the expectation value of $\hat{T}$ with respect to $|a\rangle$, we end up with

$$\begin{aligned}
\langle a|T|a\rangle &= \sum_l e^{(n-2l)^2\Delta}\left[\sum_{m=0}^{n} {}^nC_m({}^{n-m}C_l{}^mC_0 + {}^{n-m}C_{l-1}{}^mC_1e^{4K} + \ldots\right. \\
&\quad \left. + {}^{n-m}C_0{}^mC_le^{4lK})e^{(n-2l-2m)K}e^{(n-2m)^2\Delta}\right] \\
&= \int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\sum_l\left[e^{(n-2l)\sqrt{4\Delta}h_1+(n-2l)K}\left\{\sum_{m=0}^{n} {}^nC_m\left({}^{n-m}C_l{}^mC_0 + \right.\right.\right. \\
&\quad \left.\left.\left. {}^{n-m}C_{l-1}{}^mC_1e^{4K} + \ldots {}^{n-m}C_0{}^mC_le^{4lK}\right)e^{(n-2m)K+\sqrt{4\Delta}(n-2m)h_1}\right\}\right] \\
&= \int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\sum_l\left[e^{(n-2l)\sqrt{4\Delta}h_1+(n-2l)K}\sum_m^n {}^nC_mA_{mn}\right] \qquad (6.59)
\end{aligned}$$

where

$$A_{mn} = \left({}^{n-m}C_l{}^mC_0 + {}^{n-m}C_{l-1}{}^mC_1e^{4K} + \ldots {}^{n-m}C_0{}^mC_le^{4lK}\right)e^{-2mK+(n-2m)\sqrt{4\Delta}h_1}$$

To evaluate the series eq.(6.59) we consider the function

$$f_{nm}(x) = (1+x)^{n-m}(1+e^{2K}x)^me^{-2mK+(n-2m)\sqrt{4\Delta}h_1} \qquad (6.60)$$

It is easily verified that coefficient of x^l in the Taylor expansion of f_{mn} is A_{mn}. Therefore the the coefficient x^l in the sum $\sum_m^n {}^nC_mf_{mn}$ is equal to $\sum_m^n {}^nC_mA_{mn}$.But

$$\begin{aligned}
\sum_{m=0}^{m} {}^nC_mf_{mn}(x) &= \sum_{m=0}^{n} {}^nC_m(1+x)^{n-m}(1+e^{4K}x)^me^{-2mK+(n-2m)\sqrt{4\Delta}h_1} \\
&= \sum_{m=0}^{n} {}^nC_m\left\{(1+x)e^{2\sqrt{\Delta}h_1}\right\}^{n-m}\left(e^{-2K+2\sqrt{\Delta}h_1} + xe^{2K-2\sqrt{\Delta}h_1}\right)^m \\
&= \left((1+x)e^{2\sqrt{\Delta}h_1}e^{-2K-2\sqrt{\Delta}h_1} + xe^{2K-2\sqrt{\Delta}h_1}\right)^n \\
&= \left(e^{-2K-2\sqrt{\Delta}h_1} + e^{2\sqrt{\Delta}h_1} + x\left[e^{2\sqrt{\Delta}h_1} + e^{2K-2\sqrt{\Delta}h_1}\right]\right)^n \qquad (6.61)
\end{aligned}$$

Therefore the coefficient of x^l is

$$= {}^nC_l\left(e^{-2K-2\sqrt{\Delta}h_1} + e^{2\sqrt{\Delta}h_1}\right)^{n-l}\left(e^{2\sqrt{\Delta}h_1} + e^{2K-2\sqrt{\Delta}h_1}\right)^l$$

Substituting this in Eq.(6.59), we get

$$\begin{aligned}
\langle a|T|a\rangle &= \int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\left[\sum_{l=0}^{n}{}^nC_l e^{(n-2l)^2 2\sqrt{\Delta}h_2+(n-2l)K}\left(e^{-2K-2\sqrt{\Delta}h_1} + e^{2\sqrt{\Delta}h_1}\right)^{n-l}\right. \\
&\left.\left(e^{2\sqrt{\Delta}h_1} + e^{2K-2\sqrt{\Delta}h_1}\right)^l\right] \\
&= \int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\left[\sum_{l=0}^{n}{}^nC_l\left(e^{-K-2\sqrt{\Delta}h_1+2\sqrt{\Delta}h_2} + e^{K+2\sqrt{\Delta}h_1+2\sqrt{\Delta}h_2}\right)^{n-l}\right. \\
&\left.\left(e^{-K+2\sqrt{\Delta}h_1-2\sqrt{\Delta}h_2} + e^{K-2\sqrt{\Delta}h_1-2\sqrt{\Delta}h_2}\right)^l\right] \\
&= \int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\left[e^{K+2\sqrt{\Delta}(h_1+h_2)} + e^{-K+2\sqrt{\Delta}(h_1-h_2)}\right. \\
&\left. + e^{-K-2\sqrt{\Delta}(h_1-h_2)} + e^{K-2\sqrt{\Delta}(h_1+h_2)}\right]^n
\end{aligned} \tag{6.62}$$

In a similar way we obtain

$$\begin{aligned}
\langle a|a\rangle &= e^{n^2\Delta} + {}^nC_1 e^{(n-2)^2\Delta} + \ldots\ldots + {}^nC_m e^{(n-2m)^2\Delta} + \ldots\ldots \\
&= \int \frac{dh}{\sqrt{\pi}}e^{-h^2}(e^{2\sqrt{\Delta}h} + e^{-2\sqrt{\Delta}h})^n
\end{aligned} \tag{6.63}$$

We thus find

$$\lambda = \frac{\int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}(e^{K+2\sqrt{\Delta}(h_1+h_2)} + e^{-K+2\sqrt{\Delta}(h_2-h_1)} + e^{-K-2\sqrt{\Delta}(h_2-h_1)} + e^{K-2\sqrt{\Delta}(h_2+h_1)})^n}{\int \frac{dh}{\sqrt{\pi}}e^{-h^2}(e^{2\sqrt{\Delta}h} + e^{-2\sqrt{\Delta}h})^n} \tag{6.64}$$

By using the eq(6.56) and eq(6.64), we obtain the approximate free energy as

$$\begin{aligned}
F &= T\int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}}e^{-h_1^2-h_2^2}\ln(e^{K+2\sqrt{\Delta}(h_1+h_2)} + e^{-K+2\sqrt{\Delta}(h_2-h_1)} + \\
&+ e^{-K-2\sqrt{\Delta}(h_2-h_1)} + e^{K-2\sqrt{\Delta}(h_2+h_1)}) + \\
&-T\int \frac{dh}{\sqrt{\pi}}e^{-h^2}\ln(e^{2\sqrt{\Delta}h} + e^{2\sqrt{\Delta}h})
\end{aligned} \tag{6.65}$$

It is instructive to consider the free energy and entropy in the limit of $T = 0$. The free energy is found out to be

$$F = J\left(1 - \frac{2}{\pi}\int_{\frac{1}{2\omega}}^{\infty} dh e^{-h^2}\right) + \frac{4H_R}{\pi}\left(\int_{-\frac{J}{2H_R}}^{\frac{J}{2H_R}} e^{-2h^2} dh + 2e^{-\frac{J^2}{4H_R^2}} \int_{\frac{J}{2H_R}}^{\infty} e^{-h^2} dh\right) - 2\frac{H_R}{\sqrt{\pi}} \tag{6.66}$$

while entropy is zero at $T = 0$. This is in disagreement with other published results which give a finite value of entropy at $T = 0$. But it is to be noted that we are taking the distribution of the random field to be continuous. The non–zero value of entropy in the discrete model comes from the fact that the random field exsists over a set of measure zero, which is not true for the continuous distribution that we are working with.

One can improve the approximation by introducing a variational parameter α and then maximizing $\lambda(n, \alpha)$ with respect to α. This can be done by modifing the trial column matrix a. We take the trial column matrix as

$$a = \begin{pmatrix} e^{n^2\alpha\Delta/2} \\ e^{(n-2)^2\alpha\Delta/2} \\ \vdots \\ ({}^nC_1\text{elements}) \\ e^{(n-4)^2\alpha\Delta/2} \\ \vdots \\ ({}^nC_2\text{elements}) \\ e^{(n-6)^2\alpha\Delta/2} \\ \vdots \end{pmatrix} \tag{6.67}$$

The largest eigenvalue can now be estimated in a similar fashion as done before in this section. We find

$$\lambda(n, \alpha) = \frac{\int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}} e^{-h_1^2 - h_2^2} A(K, \Delta, \alpha)^n}{\int \frac{dh}{\sqrt{\pi}} e^{-h^2} B(\Delta, \alpha)^n} \tag{6.68}$$

where

$$\begin{aligned} A(K,\Delta,\alpha) &= e^{K+2\sqrt{\frac{1+\alpha}{2}\Delta}(h_1+h_2)} + e^{-K+2\sqrt{\frac{1+\alpha}{2}\Delta}(h_2-h_1)} \\ &+ e^{-K-2\sqrt{\frac{1+\alpha}{2}\Delta}(h_2-h_1)} + e^{K-2\sqrt{\frac{1+\alpha}{2}\Delta}(h_2+h_1)} \end{aligned} \tag{6.69}$$

and

$$B(\Delta,\alpha) = e^{2\sqrt{\alpha\Delta}h} + e^{-2\sqrt{\alpha\Delta}h} \tag{6.70}$$

The value $\alpha = \alpha$ that maximises $\lambda(n,\alpha)$ in the limit of $n \to 0$ is given by

$$\int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}} e^{-h_1^2-h_2^2}\frac{d}{d\alpha}\ln A(K,\Delta,\alpha) = \int \frac{dh}{\sqrt{\pi}} e^{-h^2}\frac{d}{d\alpha}\ln B(\Delta,\alpha) \tag{6.71}$$

The free energy is now easily seen to be

$$\begin{aligned} F &= T\int \frac{dh_1}{\sqrt{\pi}}\frac{dh_2}{\sqrt{\pi}} e^{-h_1^2-h_2^2}\ln A(K,\Delta,\bar{\alpha}) \\ &\quad -T\int \frac{dh}{\sqrt{\pi}} e^{-h^2}\ln B(\Delta,\bar{\alpha}) \end{aligned} \tag{6.72}$$

At $T = 0$, the expression for free energy is simplified to

$$\begin{aligned} F &= J\left(1 - \frac{2}{\pi}\int_b^\infty dhe^{-h^2}\right) \\ &\quad + \frac{2\sqrt{2}H_R\sqrt{1+\alpha}}{\pi}\left(\int_{-b}^{b} e^{-2h^2}dh + 2e^{-b^2}\int_b^\infty e^{-h^2}dh\right) - 2\frac{H_R\sqrt{\alpha}}{\sqrt{\pi}} \end{aligned} \tag{6.73}$$

where $b = \frac{J}{\sqrt{2H_R\sqrt{1+\alpha}}}$ and α is determined from

$$\sqrt{\frac{1+\alpha}{2\alpha}} = \frac{1}{\sqrt{\pi}}\left(\int_{-b}^{b} e^{-2h^2}dh + 2e^{-b^2}\int_b^\infty e^{-h^2}dh\right) \tag{6.74}$$

Appendix I

The Replica Trick

In addition to the thermal averaging one must confront the task of averaging over the random impurities in the disordered systems (For details see [53]). Basically there are two types of random impurities which are classified as annealed impurities or quenched impurities according to the way they are distributed in the host system.

Let S_i (i = 1 to N) denote the statical variable (spins in magnetism) and $\{x\}$ represent the impurity configuration. In annealed system the impurities are essentially in thermal equilibrium with the host system and so are averaged in the same manner as the statistical averages; for instance, the free energy of the system is

$$F = -kT \ln[Z\{x\}]_{av} \tag{AI.1}$$

where

$$Z\{x\} = \sum_{\{S_i\}} \exp[-H\{x, S_i\}/kT] \tag{AI.2}$$

In case of quenched impurities , the impurities in the host system are fixed while the statistical variable fluctuate. Based on the argument due to Brout [54], one can define a meaningful average whereby one finds the free energy or some other extensive variable for a given impurity configuration and then average over it. For instance the

average free energy is given by

$$\bar{F} = -kT[\ln Z\{x,S\}]_{av}$$

The fact that $\ln Z$ must be averaged rather than Z makes the problem of quenched impurities much more intractable than its annealed counterpart. To average over $\ln Z$ the unaveraged partition function has to be evaluated. Technically it is nearly impossible as one has to specify the value of typically $10^{23}4$ random variables for a macroscopic system. One way to circumvent this difficulty is to use the replica trick which is based on the identity

$$\lim_{n\to 0}\frac{x^n - 1}{n} = \ln x \tag{AI.3}$$

For positive integer n, one writes the partition function of n identical replicas of the system.

$$Z^n\{x\} = \prod_{\alpha=1}^{n} Z_\alpha\{x\} = \prod_{\alpha=1}^{n}\sum_{\{S_i^\alpha\}}\exp[-\frac{1}{kT}H(x,S_i^\alpha)] = \sum_{\{\S_i^\alpha\}}\exp[-1kT\sum_{\alpha=1}^{n}H(x,S_i^\alpha)]$$

where $Z_\alpha\{x\}$ is the partition function of the α^{th} replica.

At this stage, one averages over $Z^n\{x\}$ and expresses partition function in terms of effective hamiltonian H_{eff} which contains no disorder and is translationally invariant. The disorder in the system is manifested as some sort of an interaction between the replicas. The partition function may thus be written as

$$\langle Z^n\rangle = \sum_{\{S_i^\alpha\}}\exp[-\beta H_{eff}(n)] \tag{AI.4}$$

To be specific we consider the RFIM, where the hamiltonian is

$$H = -J\sum_{\langle ij\rangle}S_iS_j - \sum h_iS_i \tag{AI.5}$$

The partition function of n-replicas is

$$Z^n\{h_i\} = \exp[\frac{1}{T}\sum_{\alpha=1}^{n}(J\sum S_i^\alpha S_j^\alpha + \sum h_iS_i^\alpha) \tag{AI.6}$$

Averaging over the random variables $h_i's$, which we assume to have a Gaussian distribution leads to

$$\langle Z^n \rangle = \sum_{\{} S_i^\alpha \} \exp \left[\frac{J}{T} \sum_{\langle ij \rangle} S_i^\alpha S_j^\alpha + \frac{H_R^2}{T^2} \sum_{\alpha,\beta,i} S_i^\alpha S_i^\beta \right] \tag{AI.7}$$

where H_R is the random field strength.

Assuming that the analytical continuation to non-integral values of n is possible, one can evaluate the free energy by using the identity given in eq(AI.3). Therefore the free energy is

$$F = T \lim_{n \to 0} \frac{\langle Z^n \rangle - 1}{n} = T \frac{d\langle Z^n \rangle}{dn} |_{n=0} \tag{AI.8}$$

from which all other thermodynamic quantities may be evaluated.

Appendix II

Renormalisation Group For RFIM

In chapter 3, we derived renormalization group flow for the RFIM in one dimension by using the replica trick. In this appendix, we show that one gets the same structure of equations if one avoids the replicas and uses the conventional techniques where one renormalises the unaveraged Hamiltonian and keeps tracks of the probability distribution of the random field and spin coupling J.

The partition function of the random field Ising model (RFIM) is

$$Z = \sum_{\{S_i\}} \exp\left(\frac{J}{T}\sum_{(i,j)} S_i S_j + \frac{1}{T}\sum_i h_i S_i\right) \tag{AII.1}$$

where the random magnetic field has a Gaussian distribution.

$$\langle h_i h_j \rangle = \delta_{ij} H_R^2 \tag{AII.2}$$

$$\langle h_i \rangle = H \tag{AII.3}$$

We shall assume H to be infinitesimally small. As has been done in Chapters 3 and 4, we begin with $D = 1$, the Ising chain in a random magnetic field. The renormalisation procedure would be to trace over alternate spins, *i.e.*$S_2, S_4, \ldots$ etc., at various sites. If we consider the site $2i-1, 2i$, and $2i+1$, the spins at site $2i$ will be removed and

partition function will be expressed in terms of S_{2i-1} and S_{2i+1}. After this coarse-graining we replace S_{2i-1} by S_i. We thus require the relevent bit of Z which we write as

$$
\begin{aligned}
Z' &= \sum_{\{S_i\}} \exp[\frac{J}{T}(S_{2i-1}S_{2i} + S_{2i}S_{2i+1}) + \frac{1}{T}(+h_{2i}S_{2i})] && \text{(AII.4)} \\
&= \exp[\frac{J}{T}(S_{2i-1} + S_{2i+1}) + \frac{h_{2i}}{T}] + \exp[-\frac{J}{T}(S_{2i-1} + S_{2i+1}) - \frac{h_{2i}}{T}] && \text{(AII.5)} \\
&= \exp[\frac{J'}{T}(S_{2i-1}S_{2i} + S_{2i}S_{2i+1}) + \frac{h'}{T}(S_{2i-1} + S_{2i+1} + a)] && \text{(AII.6)}
\end{aligned}
$$

The three unknowns in eq.(AII.6) can be evaluated by equating eq.(AII.5) and eq.(AII.6) for three specific cases.

1. $S_{2i+1} = S_{2i-1} = 1$
2. $S_{2i+1} = S_{2i-1} = -1$
3. $S_{2i+1} = -S_{2i-1} = 1$

After a simple algebraic manipulation we find

$$
e^{4\frac{J'}{T}} = \frac{\left(e^{2\frac{J}{T}+\frac{h_2}{T}} + e^{-2\frac{J}{T}-\frac{h_2}{T}}\right)\left(e^{-2\frac{J}{T}+\frac{h_2}{T}} + e^{2\frac{J}{T}-\frac{h_2}{T}}\right)}{e^{2\frac{h_2}{T}} + e^{-2\frac{h_2}{T}}} \qquad \text{(AII.7)}
$$

$$
e^{4\frac{h'}{T}} = \frac{\left(e^{2\frac{J}{T}+\frac{h_2}{T}} + e^{-2\frac{J}{T}-\frac{h_2}{T}}\right)}{\left(e^{-2\frac{J}{T}+\frac{h_2}{T}} + e^{2\frac{J}{T}-\frac{h_2}{T}}\right)} \qquad \text{(AII.8)}
$$

We are particularly interested in the regime where $T \to 0$ and $w = \frac{H_R}{J} \ll 1$. In this limit we find (apart from exponential small corrections in w)

$$
\begin{aligned}
J_i' &= J - \frac{1}{2}|h_{2i}| \\
h' &= \frac{1}{2}h_{2i}
\end{aligned}
$$

Since we are eliminating spins at even sites, the field at odd sites is not affected. It is clear that the field at the odd sites will also contribute to the renormalisation of the random field. After some thought one gets the following expression for J' and h'.

$$J'_i = J - \frac{1}{2}|h_{2i}| \tag{AII.9}$$

$$h'_i = h_{2i-1} + \frac{1}{2}h_{2i} + \frac{1}{2}h_{2i-2} \tag{AII.10}$$

Eqs.(AII.9) and (AII.10) yield (after taking average over h_i's)

$$J' = J - \frac{1}{\sqrt{2\pi}}H_R + O(e^{-1/w^2}) \tag{AII.11}$$

$$H_R^{2'} = \frac{3}{2}H_R^2 + O(e^{-1/w^2}) \tag{AII.12}$$

$$H' = 2H + O(e^{-1/w^2}) \tag{AII.13}$$

Unfortunately, renormalisation leads to correlations between neighbouring spins. Indeed

$$\begin{aligned}\lambda' &= \langle h'_i h'_{i+1}\rangle = \langle (h_{2i-1} + \frac{1}{2}h_{2i-2} + \frac{1}{2}h_{2i})(h_{2i+1} + \frac{1}{2}h_{2i} + \frac{1}{2}h_{2i+2})\rangle \\ &= \frac{1}{4}H_R^2 + O(e^{-1/w^2})\end{aligned}$$

which was zero to start with.

The ratio

$$m = \frac{\langle h_i h_{i+1}\rangle}{\langle h_i h_i\rangle}$$

increases after each R-G iteration. A proper choice of this ratio would make it invariant under the renormalisation group transformation at the assumed fixed point $w = w^*$. This is done by taking $\langle h_i h_{i+1}\rangle = \frac{1}{4}H_R^2$ (we shall drop $O(e^{-1/w^2})$ corrections

hereafter) in the original hamiltonian. After the renormalisation, using eqs.(AII.2) and (AII.10) we get

$$\lambda' = \langle h\prime_i h\prime_{i+1} \rangle = \frac{1}{2} H_R^2 \tag{AII.14}$$

$$H_R^{2^0} = \langle h\prime_i h\prime_i \rangle = 2H_R^2 \tag{AII.15}$$

The ratio

$$m = \frac{\langle h_i h_{i+1} \rangle}{\langle h_i h_i \rangle} = \frac{1}{4}$$

remains the same after $R-G$ iteration (of course, w also changes after each $R-G$ iteration, however, if we are able to locate a fixed point in w, then $w' = w = w*$). The eq.(AII.12) is however modified to eq.(AII.15).

These relations are true in one dimension for $T \to 0$ and small w. Using Migdal - Kadanoff bond moving approximation in which we replace $J \to b^{d-1}J$, $H_R^2 \to b^{d-1}H_R^2$ and $H \to b^{d-1}H$; (there is an ambiguity in moving single spin interactions, as such we use the same procedure as described in the chapter 3) the recursion relations are modified to

$$J' = 2^{d-1}J(1 - \frac{w}{\sqrt{2\pi}}) \tag{AII.16}$$

$$H' = 2^d H \tag{AII.17}$$

$$H_R' = 2^{\frac{d}{2}} H_R \tag{AII.18}$$

If w is small the bond moving should be an excellent approximation because most of spins will be aligned in the same direction and bond moving costs small energy.

An arbitrary scale change b is given by $b = 2^N$ after N $R-G$ iterations.In the limit $b = 1 + \delta l$ we get the same structure of flow equations as discussed in chapter 3.

$$\frac{dH_R}{dl} = H_R \frac{d}{2} \tag{AII.19}$$

$$\frac{dJ}{dl} = J(d - 1 - Aw) \tag{AII.20}$$

$$\frac{dH}{dl} = Hd \tag{AII.21}$$

where A is an arbitrary constant.

Appendix III

Anisotropic Ising spin system in two dimensions

In this appendix we study a two dimensional anisotropic spin system on a rectangular lattice of N rows and M columns with spins $S_i^\alpha, i = 1, 2....M$ and $\alpha = 1, 2, ...N$ located at the intersection of i^{th} row and α^{th} column. The hamiltonian of the system is

$$H = -J\sum_{i=1}^{M}\sum_{\alpha=1}^{N} S_i^\alpha S_{i+1}^\alpha - \frac{J_l}{2N}\sum_{i=1}^{M}\sum_{\alpha,\beta=1}^{N} S_i^\alpha S_i^\beta - h\sum_{i=1}^{M}\sum_{\alpha=1}^{N} S_i^\alpha \qquad \text{(AIII.1)}$$

where J is the nearest neighbour coupling for along the rows while $\frac{J_l}{2N}$ is the weak long range coupling for spins along the columns. Though this model has nothing to do with the random fields we include it here because it has the same form as N times replicated hamiltonian of the random field and is amenable to exact treatment.

The partition function of the system is

$$Z = \sum_{\{S_i^\alpha\}} \exp\left[\frac{J}{T}\sum_{i=1}^{M}\sum_{\alpha=1}^{N} S_i^\alpha S_{i+1}^\alpha + \frac{J_l}{2NT}\sum_{i=1}^{M}\sum_{\alpha,\beta=1}^{N} S_i^\alpha S_i^\beta + \frac{h}{T}\sum_{i=1}^{M}\sum_{\alpha=1}^{N} S_i^\alpha\right] \qquad \text{(AIII.2)}$$

The model can be solved by using the important identity

$$\exp\left[\frac{J_l}{2NT}\sum_{\alpha\beta}S_i^\alpha S_i^\beta + \frac{h}{T}\sum_\alpha S_i^\alpha\right]$$

$$= \left(\frac{NJ_l}{2\pi T}\right)\int_{-\infty}^{\infty} d\lambda \exp\left[-\frac{J_l\lambda^2}{2T}\sum_{\alpha=1}^{N}(\frac{J_l\lambda+h}{T}S_i^\alpha\right]$$

The partition function may now be written as

$$Z = \left(\frac{NJ_l}{2\pi T}\right)^{N/2}\int_{-\infty}^{\infty}\prod_i^M d\lambda_i \sum_{\{S_i^\alpha\}}\exp\left[\frac{J}{T}\sum_{i=1}^{M}\sum_{\alpha=1}^{N}S_i^\alpha S_{i+1}^\alpha - N\sum_{i=1}^{M}\frac{J_l\lambda_i^2}{2T}\sum_{i=1}^{M}\sum_{\alpha=1}^{N}(\frac{J_l\lambda_i+h}{T}S_i^\alpha\right] \tag{AIII.3}$$

It is clear that '$J_l\lambda_i$' is the effective magnetic field at the i^{th} column. For a uniform system λ is expected to be a constant. Though this assumption is suggestive and intuitively obvious, we are unable to make it more rigorous. This assumption considerably simplifies our task. First of all, we evaluate the quantity

$$Z' = \sum_{\{S_i^\alpha\}}\exp\left[\frac{J}{T}\sum_{i=1}^{M}\sum_{\alpha=1}^{N}S_i^\alpha S_{i+1}^\alpha + \sum_{i=1}^{M}\sum_{\alpha=1}^{N}\frac{J_l\lambda+h}{T}S_i^\alpha\right] \tag{AIII.4}$$

This can be done by using the transfer matrix method. It is to be noted that Z' is nothing but the partition function of the N times replicated Ising model in a uniform field. Therefore

$$Z' = [\rho_1^M + \rho_2^M]^N \tag{AIII.5}$$

where ρ_1 and ρ_2 are the eigenvalues of the Ising model in a uniform magnetic field. If $\rho_1 > \rho_2$, we can neglect the contribution of ρ_2 as compared to ρ_1 in the limit of $M \to \infty$. So, finally we end up with

$$Z = \left(\frac{NJ_l}{2HT}\right)\int_{-\infty}^{\infty} d\lambda e^{-NM[\frac{J_l\lambda}{T} - \ln\rho\,]} \tag{AIII.6}$$

$$= \left(\frac{NJ_l}{2HT}\right)\int_{-\infty}^{\infty} d\lambda e^{-NM[\frac{1}{T}A(\lambda)]} \tag{AIII.7}$$

where

$$\rho_1 = e^{\frac{J}{T}}\cosh(\frac{J_L\lambda+h}{T}) + \left(e^{\frac{J}{T}}\sinh^2(\frac{J_l\lambda+h}{T}) + e^{-\frac{J}{T}}\right)$$

and

$$A(\lambda) = \frac{J_l\lambda^2}{2} - T\ln\rho_1$$

In the limit of $M \to \infty$, $N \to \infty$; the integral can easily be evaluated by the method of steepest descent. One easily finds that the free energy density is

$$f = A(\bar{\lambda}) \tag{AIII.8}$$

where $\bar{\lambda}$ is the value that minimizes $A(\lambda)$. The minimum condition

$$\frac{dA}{d\lambda}|_{\lambda=\bar{\lambda}} = 0$$

implies

$$\bar{\lambda} = \frac{\sinh(\frac{J_l\bar{\lambda}+h}{T})}{\left[\sinh^2(\frac{J_l\bar{\lambda}+h}{T}) + e^{-\frac{J}{T}}\right]^{1/2}} \tag{AIII.9}$$

The value of λ for which $A(\lambda)$ is minimized has the physical meaning of magnetization density and hence we write eq. (AIII.9) as

$$m = \frac{\sinh(\frac{J_lm+h}{T})}{\left[\sinh^2(\frac{J_lm+h}{T}) + e^{-\frac{J}{T}}\right]^1/2} \tag{AIII.10}$$

The solution of eq.(AIII.10) is most easily found by the standard graphical technique. There are three solutions for $T < T_c$ and only one for $T > T_c$. The critical temperature is given by the equation

$$\frac{J_l}{T_c} = e^{-2\frac{J}{T_c}} \tag{AIII.11}$$

Clearly m must be small for $h = 0$ and T near T_c for $T < T_c$. In this regime eq.(AIII.10) simplifies to

$$m \approx \frac{\frac{J_lm}{T} + 1/6(\frac{J_lm}{T})^3}{e^{-2\frac{J}{T}}[1 - (\frac{J_lm}{T})^2e^{4J/T}]^{1/2}} \tag{AIII.12}$$

$$\approx e^{2J/T}\frac{J_lm}{T} + 1/2(\frac{J_lm}{T})^3[e^{4\frac{J}{T}} + 1/6] \tag{AIII.13}$$

Or

$$m \sim [1 - \frac{J_l}{T}e^{2\frac{J}{T}}]^{1/2} \sim (T - T_c)^{1/2} \tag{AIII.14}$$

which gives the value of the exponent $\beta = 1/2$. One can also easily find all other exponents. These are found to have the mean-field values. The effect of the short range interaction is completely suppressed in the critical regime though the critical temperature is found be very sensitive to the changes in J as can be easily checked from eq.(AIII.11).

Bibliography

[1] Y. Imry, J. Stat. Phys. 34 , 849, 1984.

[2] T. Nattermann and J.Villain, Phase Transitions 11 , 5-52, 1989 .

[3] T. Nattermann and P. Rujan, Int. J. Mod. Phys. 3 , 1597, 1990.

[4] Y. Imry and S. K. Ma, Phys. Rev. ett. 35 , 1399, 1975.

[5] G. Grinstein, Phys. Rev. ett. 37 , 944, 1976.

[6] A. Aharony, Y. Imry and S. K. Ma , Phys. Rev. ett. 37 , 1364, 1976.

[7] A. P. Young, J. Phys. C 10 , 257 , 1977.

[8] G. Parisi and N. Sourlas, Phys. Rev. ett. 43 , 744, 1979.

[9] E.Pytte, Y.Imry and D.Mukamel, Phys. Rev. ett. 46 , 1173, 1981.

[10] K. Binder, Y. Imry, and E. Pytte, Phys. Rev. B 24 , 6736, 1981.

[11] H. S. Kogon and D. J. Wallace, J. Phys. A 14 , 257 , 1981.

[12] A. Niemi, Phy. Rev. ett. 49 , 1808, 1982.

[13] J. F. Fernandes, G. Grinstein, Y. Imry and S. Kirpatrick, Phy. Rev. ett. 51 , 203, 1983.

[14] M. Schwartz, J. Phys. C 18 , 135, 1985.

[15] M. Schwartz and A. Soffer, Phys. Rev. B 33 , 2059, 1986.

[16] A. J. Bray and M. A. Moore, J. Phys. C 21, L927, 1985.

[17] A. Berker and S. R. Mackay, Phys. Rev. B 33 , 4712, 1985.

[18] D. S. Fisher, Phys. Rev. Lett. 56 , 416, 1986.

[19] J. Villain, J. Phys.(Paris) 46 , 1843, 1985.

[20] G. Grinstein and S. K. Ma, Phs. Rev. Lett. 49 , 685, 1982.

[21] G. Grinstein and S. K. Ma, Phys. Rev.B 28 , 2588, 1983.

[22] T. Nattermann, Phys. Stat. Sol. b 131 , 563, 1985.

[23] D. S. Fisher, Phys. Rev. B 31 , 7233, 1983.

[24] D. P. Belanger, A. R. King and V. Jaccarino, Phys. Rev. B 31 , 4538, 1985.

[25] D. P. Belanger, A. R. King and V. Jaccarino, Phys. Rev. B 34 , 452, 1986.

[26] A. R. King, J. A. Mydosh and V. Jaccarino, Phys. Rev. Lett. 56 , 2525, 1986.

[27] D. P. Belanger, V. Jaccarino, A. R. King and R. M. Nicklow, Phys. Rev. Lett. 59 , 930, 1987.

[28] A. E. Nash, A. R. King and V. Jaccarino, Phys. Rev. B 43 , 1272, 1990.

[29] S. Geschwind, David A. Huse and G. E. Devlin, Phys. Rev. B 41 , 4854, 1989.

[30] D. Boyanovsky and J. L. Cardy, Phys. Rev. B 27 , 557, 1983.

[31] U. Krey and H. Ostermayer, . Physik B 66 , 1219, 1988.

[32] B. Caroli, C. Caroli, B. Roult and J. F. Gonyet, J. Stat. Phys. 22 , 515, 1980.

[33] J. K. Bhattacharjee and K. Banerjee, J. Phys. A : Math Gen. 22 , L1141, 1989.

[34] R. A. Ferrell and D. J. Scalpino, Phys. Rev. A 9 , 846, 1974.

[35] R. Balian and G. Toulouse, Annals of Phys.83,28,1974.

[36] A. J. Bray and Mackane, Euro. Phys. Lett. 1, 427, 1986.

[37] W. E. Caswell, Ann. Phys.(N.Y.) 123, 153, 1987.

[38] K. Banerjee and J. K. Bhattacharjee, Phys. rev. D 29, 1111, 1984.

[39] A. A. Migdal, Sov. Phys. JETP 42, 743, 1976.

[40] L. P. Kadanoff, Ann. Phys. (N.Y.) 100, 359, 1976.

[41] T. W. Burkhardt: In Real Space Renormalization,ed. by T. N. Burkhardt and J. M. J. van Leeuwen,(Springer, Berlin, Heidelberg, New York)

[42] S. K. Ma, Modern Theory of Critical Phenomena, (Benjamin, Reading 1976).

[43] G. Parisi: Statisical Field Theory, (Addision-Wesley, 1988).

[44] S. Fishman and A. Aharony, J. Phys. C 12, L729, 1979.

[45] J. Imbrie, Phys. Rev. Lett. 53, 1747, 1983.

[46] G.Grinstein and D. Mukamel, Phys. Rev. B 27, 4507, 1983.

[47] B. Derrida and H. J. Hilhorst, J. Phys. C 14, L539, 1981.

[48] G. Gyorgyi and P. Rujan, J. Phys. C 17, 4207, 1984.

[49] Th. M. Nieuwenhuizen and J. M. Luck, J. Phys. A 19, 1207, 1986.

[50] K.Binder, .Phys. B 50, 343, 1983.

[51] H. E. Stanley, Introduction to phase transitions and critical phenomena, Clarendon Press, Oxford, 1971.

[52] A. D. Bruce and D. J. Wallace, J. Phys. A: math. Gen. 16, 1721, 1983.

[53] K. Binder and A. P. Young, Rev. Mod. Phys. 58, 801, 1986.

[54] R. Brout, Phys. Rev. 115, 824, 1959.

PHY-1993-D-PAR-CRI